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Calculus

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Chapter 1

Building Blocks of Calculus

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Section 1

Precalculus Review

Real Numbers, Functions, and Graphs

Linear and Quadratic Functions

Trigonometric Functions

Inverse Functions

Exponential and Logarithmic Functions

Using Calculators and Computers

https://www.boundless.com/calculus/building-blocks-of-calculus/precalculus-review/

Real Numbers, Functions, and Graphs

Functions relate a set of inputs to a distinct output. Graphs can be pictorial representations of these relationships.

KEY POINTS

- The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as √2 (1.41421356… the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number).
- A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.
- The graph of a function f is the collection of all ordered pairs (x, f(x)).

Real Numbers

Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as √2 (1.41421356… the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number).

Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one. The real line can be thought of as a part of the complex plane, and correspondingly, complex numbers include real numbers as a special case.

Functions

A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number x to its square x²: $f(x) = x^2$ (x: real number). Here, the **domain** is the entire set of real numbers and the function maps each real number to the square of it. The output of a function f corresponding to an input x is denoted by f(x) (read "f of x"). In this example, if the input is −3, then the output is 9, and we may write f(−3) = 9. The input variable(s) are sometimes referred to as the argument(s) of the function.

Graphs

The graph of a function f is the collection of all ordered pairs (x, f(x)). In particular, if x is a real number, graph means the graphical representation of this collection, in the form of a line chart, a curve on a **Cartesian** plane, together with Cartesian axes, etc. Graphing on a Cartesian plane is sometimes referred to as curve sketching. If the function input x is an ordered pair (x_1, x_2) of real numbers, the graph is the collection of all ordered triples $(x_1, x_2, f(x_1, x_2))$, and its graphical representation is a surface (see three dimensional graph) (*Figure 1.1*).

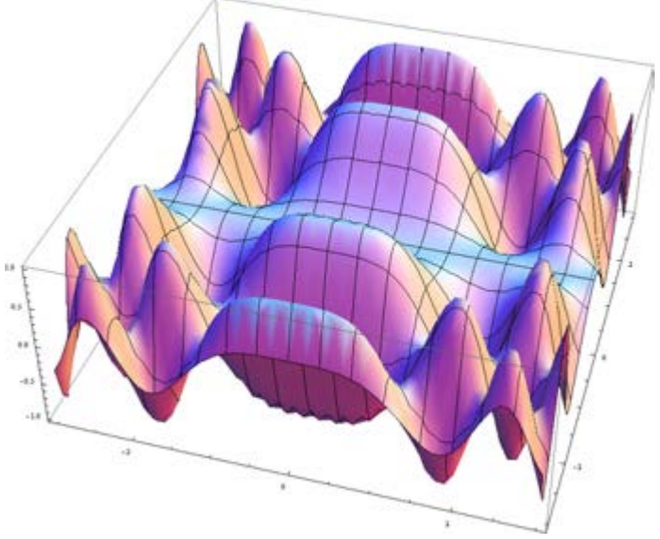

Figure 1.1 Graph of a Function

This is a graph of the function $f(x, y) = sin(x^2)cos(y^2)$

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Linear and Quadratic Functions

Linear and quadratic functions make lines and a parabola, respectively, when graphed and are some of the simplest functional forms.

KEY POINTS

- Linear function refers to a function that satisfies the following two linearity properties:
  $f(x+y) = f(x) + f(y)$
  $f(ax) = af(x)$
- Linear functions may be confused with affine functions. One variable affine functions can be written as f(x) = mx + b, which makes a line when graphed.
- A quadratic function, in mathematics, is a polynomial function of the form: $f(x) = ax^2 + bx + c, \quad a \neq 0$. The graph of a quadratic function is a parabola whose axis of symmetry is parallel to the y-axis.

Linear and quadratic functions make lines and parabola, respectively, when graphed. They are one of the simplest functional forms.

Linear Function

In calculus and algebra, the term linear function refers to a function that satisfies the following two **linearity** properties:

$f(x+y) = f(x) + f(y)$
$f(ax) = af(x)$

Linear functions may be confused with affine functions. One variable affine functions can be written as f(x) = mx + b. Although affine functions make lines when graphed, they do not satisfy the properties of linearity. However, the term "linear function" is quite often loosely used to include affine functions of the form f(x) = mx + b. Linear functions form the basis of linear algebra.

Affine Function

An affine transformation (from the Latin, affinis, "connected with") is a transformation which preserves straight lines (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances between points lying on a straight line (e.g., the midpoint of a line segment remains the midpoint after transformation). It does not necessarily preserve angles or lengths, but does have the property that sets of parallel lines will remain parallel to each other after an affine transformation.

Quadratic Function

A quadratic function, in mathematics, is a **polynomial** function of the form: $f(x) = ax^2 + bx + c, a \neq 0$. The graph of a quadratic function is a parabola whose axis of symmetry is parallel to the y-axis (*Figure 1.2*). The expression $ax^2 + bx + c$ in the definition of a quadratic function is a polynomial of degree 2 or second order, or a 2nd degree polynomial, because the highest exponent of x is 2. If the quadratic function is set equal to zero, then the result is a quadratic equation. The solutions to the equation are called the roots of the equation.

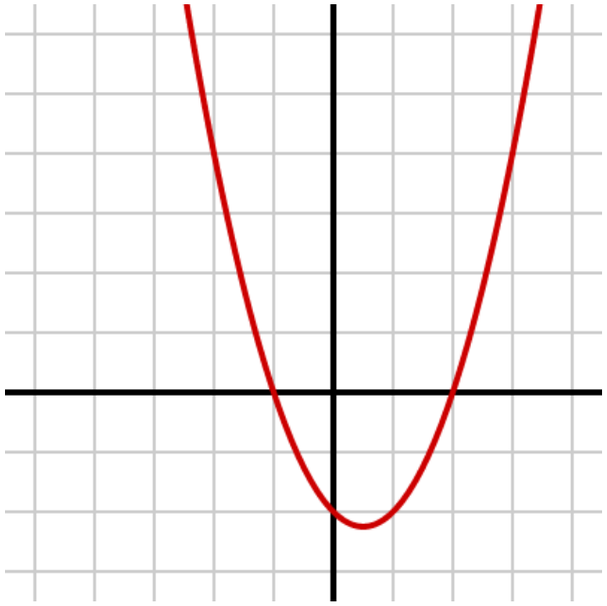

Figure 1.2
Quadratic Function
Graph of a quadratic function $f(x) = x^2 - x - 2$. It has a shape of a parabola.

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Trigonometric Functions

Trigonometric functions are functions of an angle, relating the angles of a triangle to the lengths of the sides of a triangle.

KEY POINTS

- The most familiar trigonometric functions are the sine, cosine, and tangent.
- The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse.
- The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse.
- The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side.

Trigonometric functions are functions of an angle. They are used to relate the angles of a triangle to the lengths of the sides of a triangle. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.

The most familiar trigonometric functions are the sine, cosine, and tangent (*Figure 1.3*). In the context of the standard unit circle with radius 1, where a triangle is formed by a ray originating at the origin and making some angle with the x-axis, the sine of the angle gives

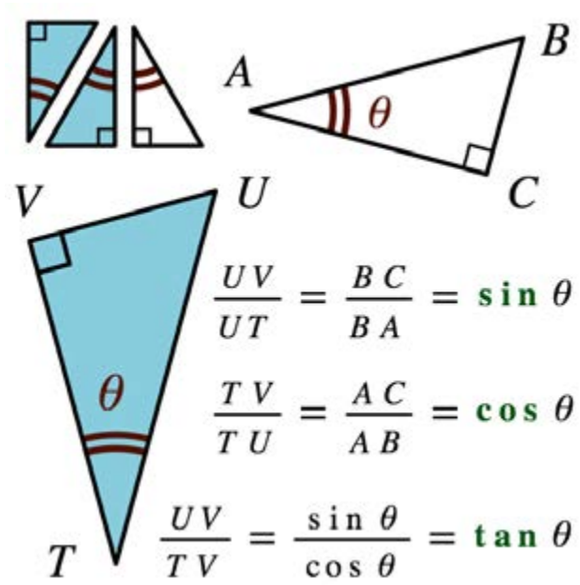

Figure 1.3 Base of Trigonometry

If two right triangles have equal acute angles, they are similar, so their side lengths are proportional. Proportionality constants are written within the image: sin θ, cos θ, tan θ, where θ is the common measure of five acute angles.

the length of the y-component (rise) of the triangle, the cosine gives the length of the x-component (run), and the tangent function gives the slope (y-component divided by the x-component) (*Figure 1.4*).

Trigonometric functions are commonly defined as ratios of two sides of a right triangle containing the angle, and can equivalently be defined as the lengths of various line segments from

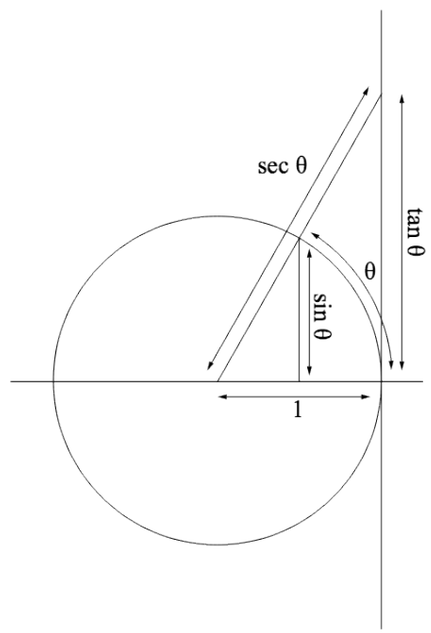

Figure 1.4 Sine, Tangent, and Secant

The sine, tangent, and secant (=1/cos) functions of an angle constructed geometrically in terms of a unit circle. The number θ is the length of the curve; thus angles are being measured in radians.

a unit circle. More modern definitions express them as infinite series or as solutions of certain differential equations, allowing their extension to arbitrary positive and negative values and even to complex numbers.

Right-Angle Triangle Definitions

The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. In our case $\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{h}$ (*Figure 1.5*). Note that this ratio does not depend on size of the particular right triangle chosen, as long as it contains the angle A, since all such triangles are similar.

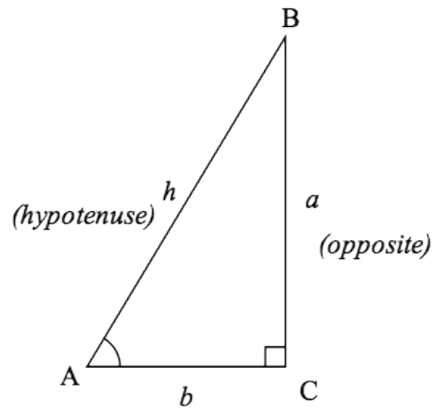

Figure 1.5 Right-Angle Triangle

Right-angle triangle used in defining trigonometric functions.

The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse: so called because it is the sine of the complementary or co-angle. In our case $\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{h}$.

The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side: so called because it can be represented as a line segment tangent to the circle, that is the line that touches the circle, from Latin linea tangens or touching line (cf. tangere, to touch). In our case $\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$.

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Inverse Functions

Inverse function is a function that undoes another function:For a function f(x)=y, inverse function, if exists, is given as g(y)=x.

KEY POINTS

- A function f that has an inverse is called invertible; the inverse function is then uniquely determined by f and is denoted by f^{-1}.
- If f is invertible, the function g is unique.
- The function $f(x) = x^2$ may or may not be invertible, depending on the domain. For a domain containing all real numbers, it is not invertible. But if the domain consists of the non-negative numbers, then the function is injective and invertible.

Inverse function is a function that undoes another function: If an input x into the function f produces an output y, then putting y into the inverse function g produces the output x, and vice versa. i.e., f(x)=y, and g(y)=x. More directly, g(f(x))=x, meaning g(x) composed with f(x) leaves x unchanged.

A function f that has an inverse is called invertible; the inverse function is then uniquely determined by f and is denoted by f^{-1} (read f inverse, not to be confused with exponentiation, *Figure 1.6*).

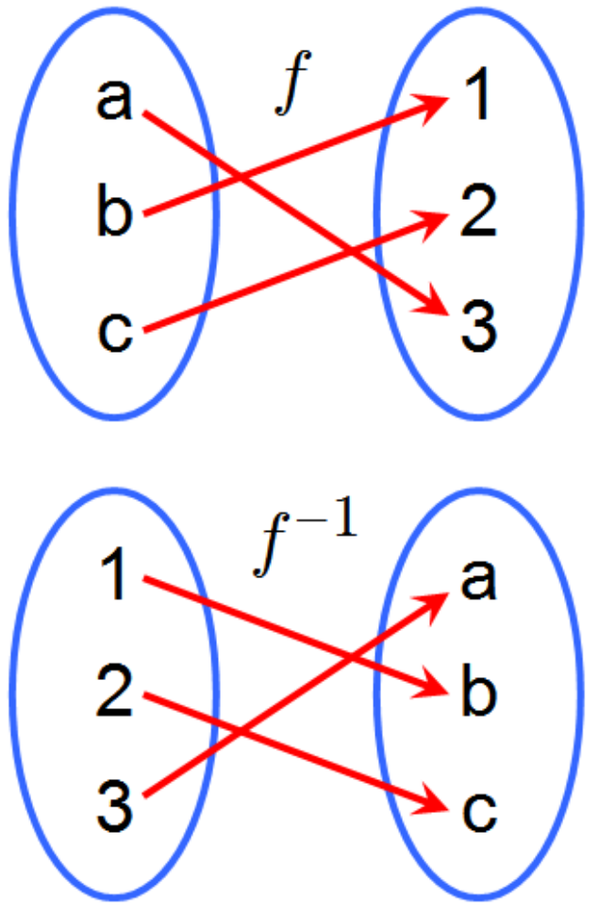

Figure 1.6 A Function and its Inverse

A function f and its inverse f^{-1}. Because f maps a to 3, the inverse f^{-1} maps 3 back to a.

If f is invertible, the function g is unique; in other words, there is exactly one function g satisfying this property (no more, no less). Not all functions have an inverse. For this rule to be applicable, for a function whose domain is the set X and whose **range** is the set Y, each element y ∈ Y must correspond to no more than one x ∈ X; a function f with this property is called one-to-one, or information-preserving, or an injection.

EXAMPLES

Inverse operations that lead to inverse functions

Inverse operations are the opposite of direct variation functions. Direct variation function are based on multiplication; y = kx. The opposite operation of multiplication is division and an inverse variation function is y = k/x.

Squaring and square root functions

The function $f(x) = x^2$ may or may not be invertible, depending on the domain. If the domain is the real numbers, then each element in the range Y would correspond to two different elements in the domain X (±x), and therefore f would not be invertible. More precisely, the square of x is not invertible because it is impossible to deduce from its output the sign of its input. Such a function is called non-injective or information-losing. Notice that neither the square root nor the principal square root function is the inverse of x^2 because the first is not single-valued, and the second returns -x when x is negative. If the domain consists of the non-negative numbers, then the function is injective and invertible.

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Exponential and Logarithmic Functions

Both exponential and logarithmic functions are widely used in scientific and engineering applications.

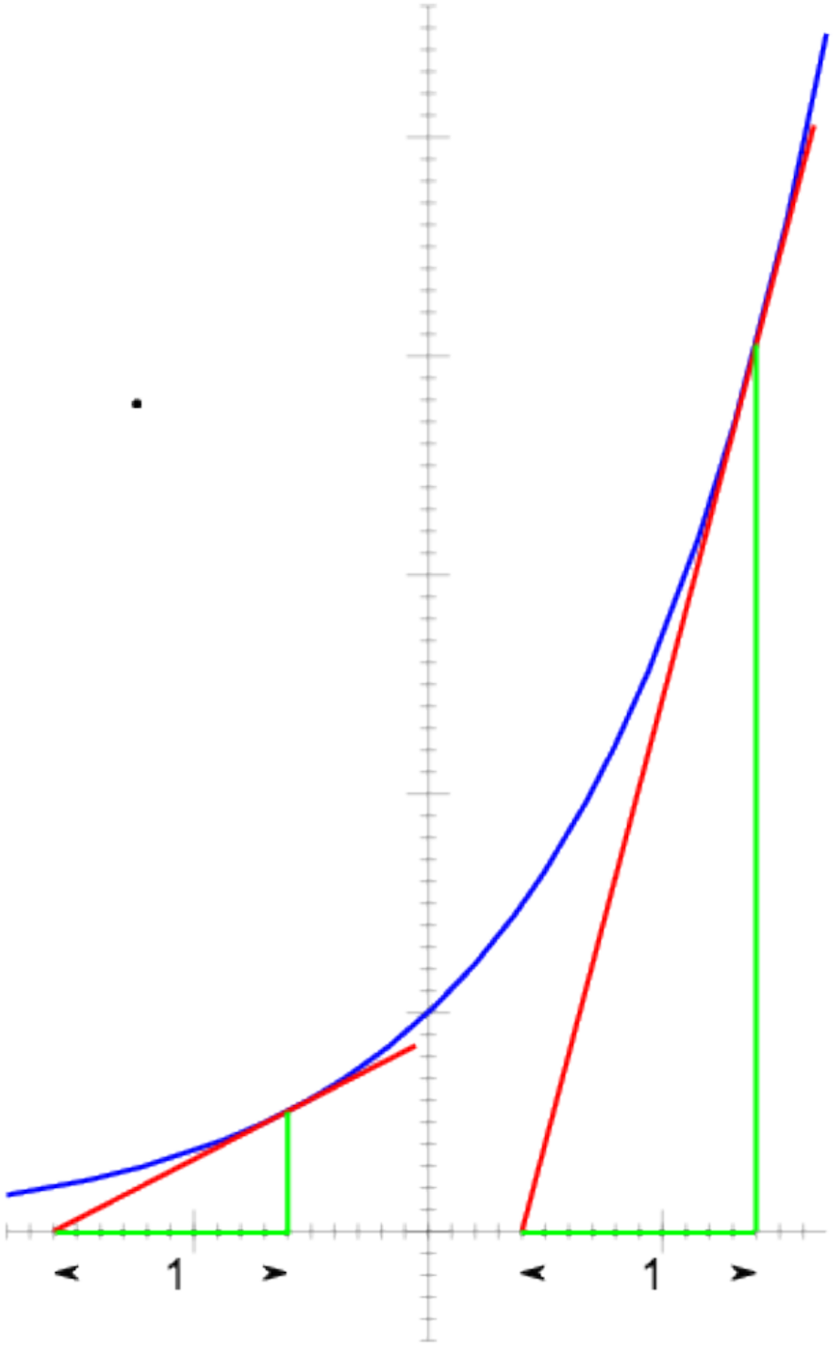
KEY POINTS

- Exponential function is the function e^x, where e is the number (approximately 2.718281828) such that the function e^x is its own derivative.
- The exponential function may be defined by the following power series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- The logarithm of a number x with respect to base b is the exponent by which b must be raised to yield x. In other words, the logarithm of x to base b is the solution y to the equation $b^y = x$.

Exponential Function

Exponential function is the function e^x, where e is the number (approximately 2.718281828) such that the function e^x is its own derivative (*Figure 1.7*). The function is often written as exp(x), especially when it is impractical to write the independent variable as a superscript. The exponential function is widely used in physics,

Figure 1.7 Graph of $f(x) = e^x$



The derivative (or slope of a tangential line) of the exponential function is equal to the value of the function.

chemistry, engineering, mathematical biology, economics and mathematics. Sometimes the term exponential function is used more generally for functions of the form $f(x) = cb^x$, where the base b is any positive real number, not necessarily e and c is a constant.

The exponential function arises whenever a quantity grows or decays at a rate proportional to its current value. One such situation is continuously compounded interest, and in fact it was this that led Jacob Bernoulli in 1683 to the number $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, now known as e. Similarly, $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ first given by Euler.

The exponential function e^x can be characterized in a variety of equivalent ways. In particular it may be defined by the following power series: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. From this definition, you can check that e^x is its own derivative: $\frac{d}{dx}e^x = e^x$.

Logarithmic Functions

The logarithm of a number x with respect to base b is the exponent by which b must be raised to yield x. In other words, the logarithm of x to base b is the solution y to the equation $b^y = x$ (*Figure 1.8*). For example, the logarithm of 1000 to base 10 is 3, because 1000 is

10 to the power 3: $1000 = 10 \cdot 10 \cdot 10 = 103$. More generally, if $x = b^y$, then y is the logarithm of x to base b, and is written $y = \log_b(x)$, so $\log_{10}(1000) = 3$.

For $f(x)=e^x$, $g(x)=\log_e x$ is the inverse function of f(x) and vice versa. The logarithm to base b = 10 is called the common logarithm and has many applications in science and engineering. The natural logarithm has the constant e (≈ 2.718) as its base; its use is widespread in pure mathematics, especially calculus. The binary logarithm uses base b = 2 and is prominent in computer science.

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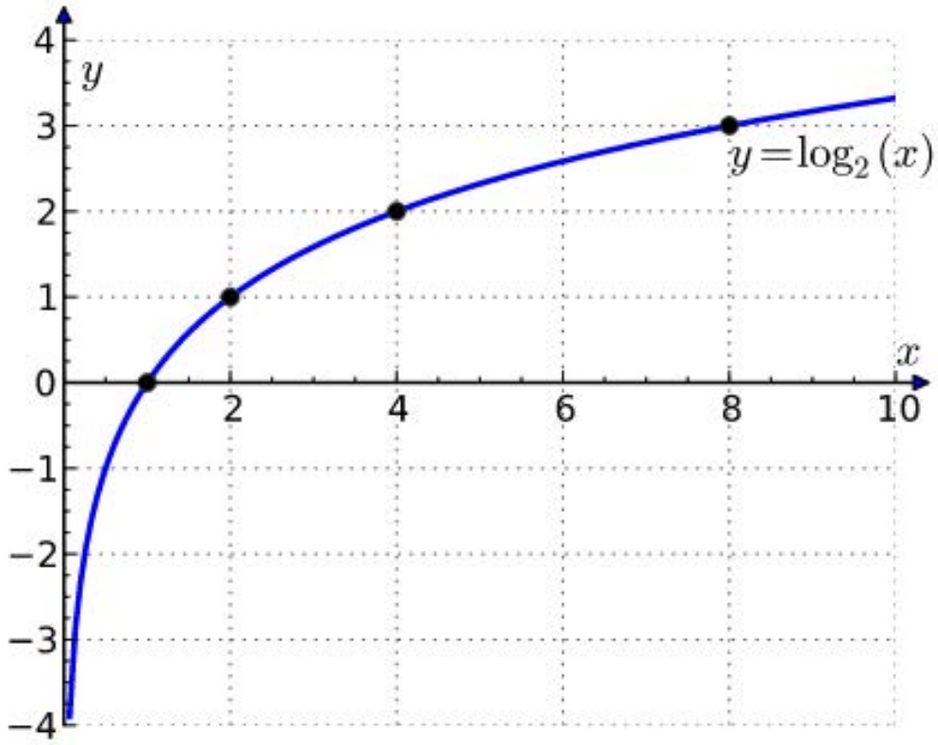
Figure 1.8 Graph of Log Base 2



The graph of the logarithm to base 2 crosses the x axis (horizontal axis) at 1 and passes through the points with coordinates (2, 1), (4, 2), and (8, 3). For example, $\log_2(8) = 3$, because $2^3 = 8$. The graph gets arbitrarily close to the y axis, but does not meet or intersect it.

Using Calculators and Computers

For numerical calculations and graphing, scientific calculators and personal computers are commonly used in classes and laboratories.

KEY POINTS

- A scientific calculator is a type of electronic calculator, usually but not always handheld, designed to calculate problems in science, engineering, and mathematics.
- These days, scientific and graphing calculators are often replaced by personal computers or even by supercomputers.
- Either by using commercial softwares or by using programming languages, complicated, multi-step numerical calculations can be performed on a PC.

Scientific calculators

A scientific calculator is a type of electronic calculator, usually but not always handheld, designed to calculate problems in science, engineering, and mathematics. They have almost completely replaced slide rules in almost all traditional applications, and are widely used in both education and professional settings.

In certain contexts such as higher education, scientific calculators have been superseded by graphing calculators (*Figure 1.9*), which offer a superset of scientific calculator functionality along with the ability to graph input data and write and store programs for the device. There is also some overlap with the financial calculator market.

Figure 1.9
A typical graphing calculator by Texas Instruments, displaying a graph of a function $f(x) = 2x^{2-3}$.

Scientific calculators are used widely in any situation where quick access to certain mathematical functions is needed, especially those such as trigonometric functions that were once traditionally looked up in tables; they are also used in situations requiring back-of-the-envelope calculations of very large numbers, as in some aspects of astronomy, physics, and chemistry. However, for more complicated applications, computers offer more powerful solutions to many problems.

Computers

These days, scientific and graphing calculators are often replaced by personal computers or even by **supercomputers**. There are many softwares (such as Mathematica, Matlab, etc.) that allows not only

numerical calculations and plotting of functions, but also helping with matrix manipulations, data manipulation, implementation of algorithms, creation of user interfaces, etc. (*Figure 1.10*) In addition, by using **programming languages** such as Fortran, C, C++, Java, etc., complicated, multi-step numerical calculations can be performed on a PC. Therefore, computers are extremely useful tools in scientific and engineering disciplines.

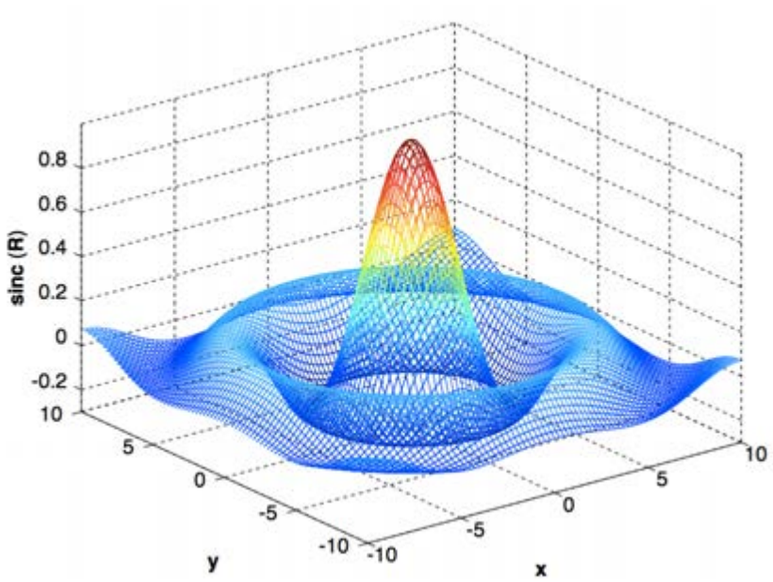


Figure 1.10

A three-dimensional wireframe plot of the unnormalized sinc function $z = sinc[(x^2 + y^2)^{0.5}]$

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Section 2

Functions and Models

Four Ways to Represent a Function

Essential Functions for Mathematical Modeling

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Four Ways to Represent a Function

Functions can be expressed in four different ways.

KEY POINTS

- A function can be represented verbally. For example, the circumference of a square is four times one of its sides.
- A function can be represented algebraically. For example, 3x + 6.
- A function can be represented numerically.
- A function can be represented graphically.

A **function** is a relation between a set of inputs and a set of permissible outputs, provided that each input is related to exactly one output. An example is the function that relates each real number x to its square x^2. The output of a function f corresponding to an input x is denoted by f(x) (read "f of x"). In this example, if the input is −3, then the output is 9, and we may write f(−3) = 9. The input variable(s) are sometimes referred to as the argument(s) of the function.

Modern calculus texts emphasize that a function can be expressed in four different ways.

Verbal: When modeling a process mathematically, one often first develops a verbal description of the problem. For example, the expression 2x + 6 can be represented as “Double x and add six” or "six added to two times x".

Algebraic: This is the most common, most concise, and most powerful representation: 2x+6. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).

Numerical: This can be expressed as a list of value pairs, as in (4,14) — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as the (x,y) points used in graphing.)

Graphical: This involves modeling a function in a dimensional overlay. Scientific data is often recorded in a visual format. Examples include seismograph readings, electrocardiograms, and oscilloscope readings (*Figure 1.11*).

These are not four different types of functions; they are four different views of the same function. One of the most important skills in algebra and calculus is being able to convert a function between these different forms, and this theme will recur in different forms throughout the text.

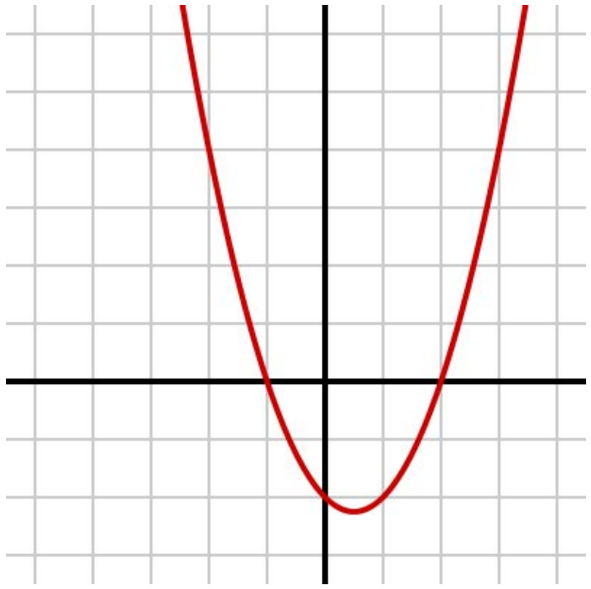

Figure 1.11 Graphical Representation of a Function

A function is represented by a two-dimensional graph.

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Essential Functions for Mathematical Modeling

Mathematical models are used to explain system, study effects of components, and make predictions about behavior.

KEY POINTS

- A mathematical model is a description of a system using mathematical concepts and language.
- Mathematical models are used in the natural sciences, engineering disciplines, and in the social sciences.
- Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models.

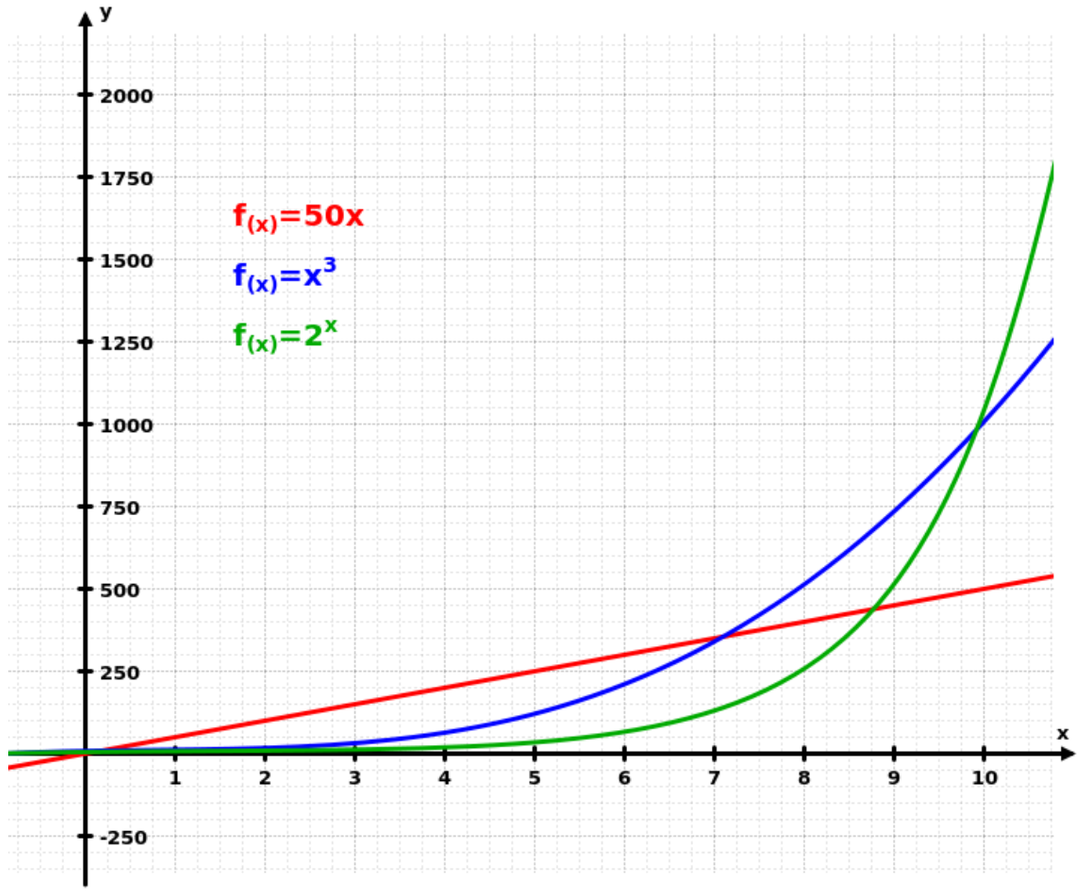
A mathematical model is a description of a system using mathematical concepts and language. The process of developing a **mathematical model** is termed mathematical modeling. Mathematical models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science); physicists, engineers,

statisticians, operations research analysts and economists use mathematical models most extensively. A model may help to explain a system and to study the effects of different components, and to make predictions about behavior.

Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed.

For example, a simple model of population growth is the Malthusian growth model. This is essentially **exponential growth** (*Figure 1.12*) based on a constant rate of compound interest: $P(t) = P_0 e^{rt}$ where P_0 = P(0) = initial population, r = growth rate, and, t is the time. A slightly more realistic and largely used population growth model is the logistic function (see *Figure 1.13*) that may be defined by the formula: $P(t) = \frac{1}{1 + e^{-t}}$.

Figure 1.12 Exponential Growth



The graph illustrates how exponential growth (green) surpasses both linear (red) and cubic (blue) growth.

$$P(t) = \frac{1}{1 + e^{-t}}$$

Another example is a model of a particle in a potential-field. In this model we consider a particle as being a point of mass which describes a trajectory in space which is modeled by a function giving its coordinates in space as a function of time. The potential field is

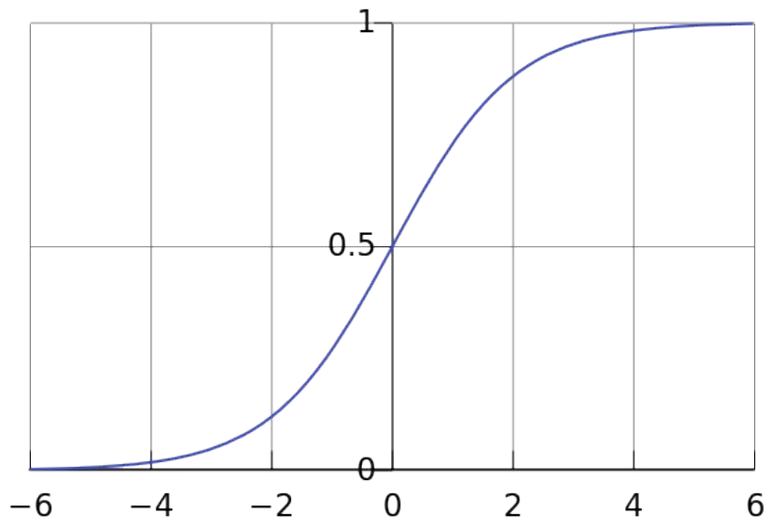


Figure 1.13
Logistic Function
Standard logistic sigmoid function

given by a function V : $R^3 \to$ R and the trajectory is a solution of a differential equation.

Even many everyday activities carried out without a thought are uses of mathematical models. A geographical **map projection** of a region of the earth onto a small, plane surface is a model which can be used for many purposes such as planning travel.

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Section 3

Limits

Tangent and Velocity Problems

Limit of a Function

Calculating Limits Using the Limit Laws

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Tangent and Velocity Problems

Instantaneous velocity can be obtained from a position-time curve of a moving object.

KEY POINTS

- Velocity is defined as rate of change of displacement.
- The velocity v of the object can be computed as the derivative of position: $v = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$
- The equation for an object's position can be obtained by evaluating the integral of the equation for its velocity from time t_0 to a later time t_n.

Calculus has widely used in physics and engineering. In this atom, we will learn that instantaneous velocity can be obtained from a position-time curve of a moving object by calculating derivatives of the curve.

Velocity is defined as rate of change of displacement. The average velocity $\bar{v}$ of an object moving through a displacement (Δx) during a time interval (Δt) is described by the formula: $\bar{v} = \frac{\Delta x}{\Delta t}$.

What will happen when we reduce the time interval Δt and let it approach 0? The average velocity becomes instantaneous velocity at time t. Suppose an object is at positions x(t) at time t and $x(t + \Delta t)$ at time $t + \Delta t$. The velocity v of the object can be computed as the derivative of position: $v = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$ (*Figure 1.14*).

Instantaneous velocity is always tangential to trajectory. Slope of tangent of position or displacement time graph is instantaneous velocity and its slope of chord is average velocity.

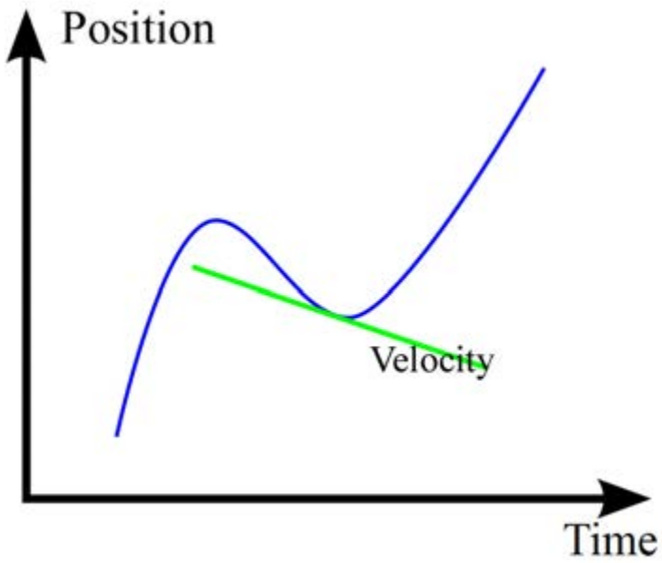

Figure 1.14
Instantaneous Velocity
The green line shows the tangential line of the position-time curve at a particular time. Its slope is the velocity at that point.

On the other hand, the equation for an object's position can be obtained mathematically by evaluating the definite integral of the equation for its velocity beginning from some initial period time to to some point in time later t_n. That is $x(t) = x_0 + \int_{t_0}^{t} v(t')\, dt'$, where xo

is the position of the object at t = t_0. For the simple case of constant velocity, the equation gives $x(t) - x_0 = v_0(t - t_0)$.

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Limit of a Function

The limit of a function is a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input.

KEY POINTS

- The function has a limit L at an input p if f(x) is "close" to L whenever x is "close" to p.
- To say that $\lim_{x \to p} f(x) = L$ means that f(x) can be made as close as desired to L by making x close enough, but not equal, to p.
- For x approaching p from above (right) and below (left), if both of these limits are equal to L then this can be referred to as the limit of f(x) at p.

The limit of a **function** is a fundamental concept in calculus and analysis concerning the behavior of that function near a particular input. Informally, a function f assigns an output f(x) to every input x. The function has a limit L at an input p if f(x) is "close" to L whenever x is "close" to p. In other words, f(x) becomes closer and closer to L as x moves closer and closer to p. More specifically, when f is applied to each input sufficiently close to p, the result is an output value that is arbitrarily close to L. If the inputs "close" to p

are taken to values that are very different, the limit is said to not exist.

The notion of a limit has many applications in modern calculus. In particular, the many definitions of continuity employ the limit: roughly, a function is continuous if all of its limits agree with the values of the function. It also appears in the definition of the **derivative**: in the calculus of one variable, this is the limiting value of the slope of **secant lines** to the graph of a function.

To say that $\lim_{x \to p} f(x) = L$ means that f(x) can be made as close as desired to L by making x close enough, but not equal, to p.

Alternatively, x may approach p from above (right) or below (left), in which case the limits may be written as $\lim_{x \to p^+} f(x) = L_+$ or $\lim_{x \to p^-} f(x) = L\_$. If both of these limits are equal to L then this can be referred to as the limit of f(x) at p. Conversely, if they are not both equal to L then the limit, as such, does not exist (*Figure 1.15*). In the following atoms, we will learn about more strict and precise definition of the limit of a function.

Example: A function without a limit: $f(x) = \begin{cases} \sin \frac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \frac{0.1}{x-1} & \text{for } x > 1 \end{cases}$ has

has no limit at $x_0 = 1$.

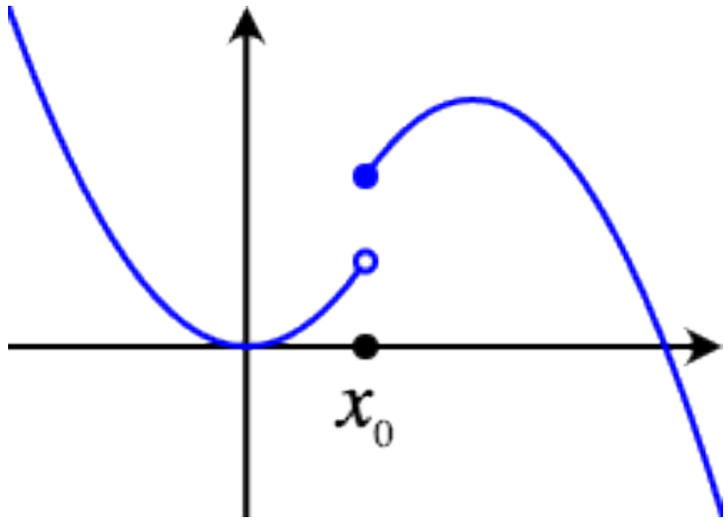

Figure 1.15 The Limit Does Not Exist

The limit as: x → x0+ ≠ x → x0-. Therefore, the limit as x → x0 does not exist.

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Calculating Limits Using the Limit Laws

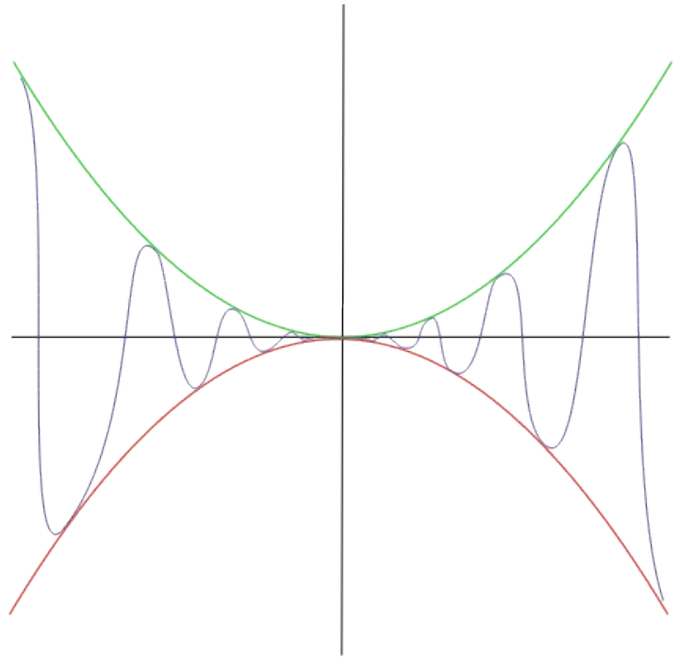
Limits of functions can often be determined using simple laws, such as L'Hôpital's rule and squeeze theorem.

KEY POINTS

- L'Hôpital's rule uses derivatives to help evaluate limits involving indeterminate forms.
- When using the L'Hôpital's rule, the differentiation of the numerator and denominator often simplifies the quotient and/or converts it to a determinate form, allowing the limit to be evaluated more easily.
- The squeeze theorem is often used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

Limits of functions can often be determined using simple laws. In this atom, we will study two examples: L'Hôpital's rule or the Squeeze theorem.

Figure 1.16 Squeeze Theorem



$x^{2sin(1/x)}$ being squeezed by x^2 and - x^2 in the limit as x goes to 0.

L'Hôpital's Rule

L'Hôpital's rule (pronounced: [lopi'tal], sometimes spelled l'Hospital's rule with silent "s" and identical pronunciation), also called Bernoulli's rule, uses **derivatives** to help evaluate limits involving indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to a determinate form, allowing easy evaluation of the limit.

In its simplest form, l'Hôpital's rule states that for functions f and g which are **differentiable** on I{c}, where I is an open interval

containing c: $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm \infty$, and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x in I (x ≠ c), then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$. The differentiation of the numerator and denominator often simplifies the quotient and/or converts it to a determinate form, allowing the limit to be evaluated more easily.

Squeeze Theorem

The squeeze theorem is a technical result that is very important in proofs in calculus and mathematical analysis. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed (*Figure 1.16*). The squeeze theorem is formally stated as follows.

Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have $g(x) \leq f(x) \leq h(x)$, and also suppose that $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

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Precise Definition of a Limit

The (ε, δ)-definition of limit ("epsilon-delta definition of limit") is a formalization of the notion of limit.

KEY POINTS

- Suppose f : R → R is defined on the real line and p,L ∈ R. It is said the limit of f as x approaches p is L and written $\lim_{x \to p} f(x) = L$, if the following property holds.
- For every real ε > 0, there exists a real δ > 0 such that for all real x, 0 < | x − p | < δ implies | f(x) − L | < ε. (*Figure 1.17*) Note that the value of the limit does not depend on the value of f(p), nor even that p be in the domain of f.
- This definition also works for functions with more than one input value.

The (ε, δ)-definition of limit ("epsilon-delta definition of limit") is a formalization of the notion of limit. It was first given by Bernard Bolzano in 1817, followed by a less precise form by Augustin-Louis Cauchy. The definitive modern statement was ultimately provided by Karl Weierstrass.

(ε,δ)-Definition

(ε, δ)-definition of limit is a formalization of the notion of limit. Suppose f : R → R is defined on the real line and p,L ∈ R. It is said the limit of f as x approaches p is L and written $\lim_{x \to p} f(x) = L$, if the following property holds: For every real ε > 0, there exists a real δ > 0 such that for all real x, 0 < | x − p | < δ implies | f(x) − L | < ε. (*Figure 1.17*) Note that the value of the limit does not depend on the value of f(p), nor even that p be in the domain of f.

The letters ε and δ can be understood as "**error**" and "distance", and in fact Cauchy used ε as an abbreviation for "error" in some of his work. In these terms, the error (ε) in the measurement of the value at the limit can be made as small as desired by reducing the distance (δ) to the limit point.

This definition also works for functions with more than one input value. In those cases, δ can be understood as the radius of a circle or sphere or higher-dimensional analogy, in the domain of the function and centered at the point where the existence of a limit is being proven, for which every point inside produces a function value less than ε from the value of the function at the limit point.

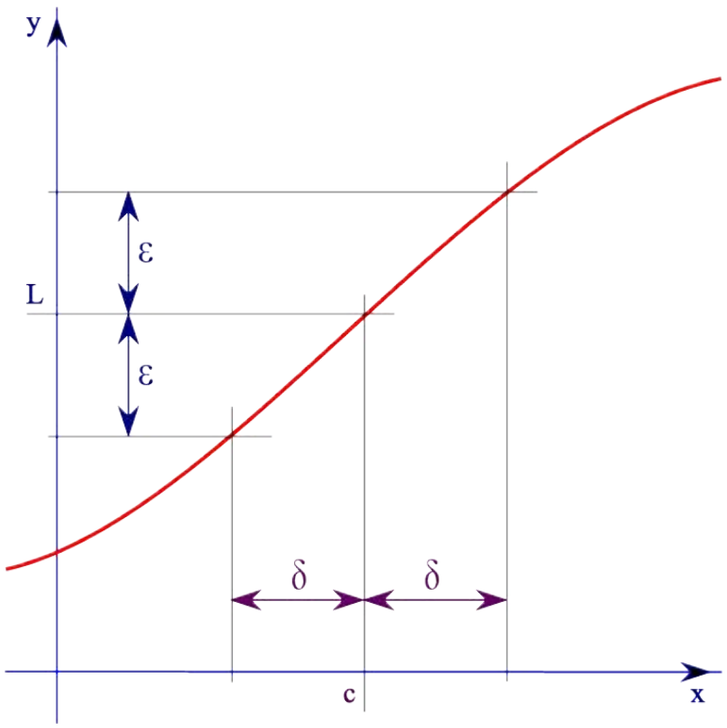
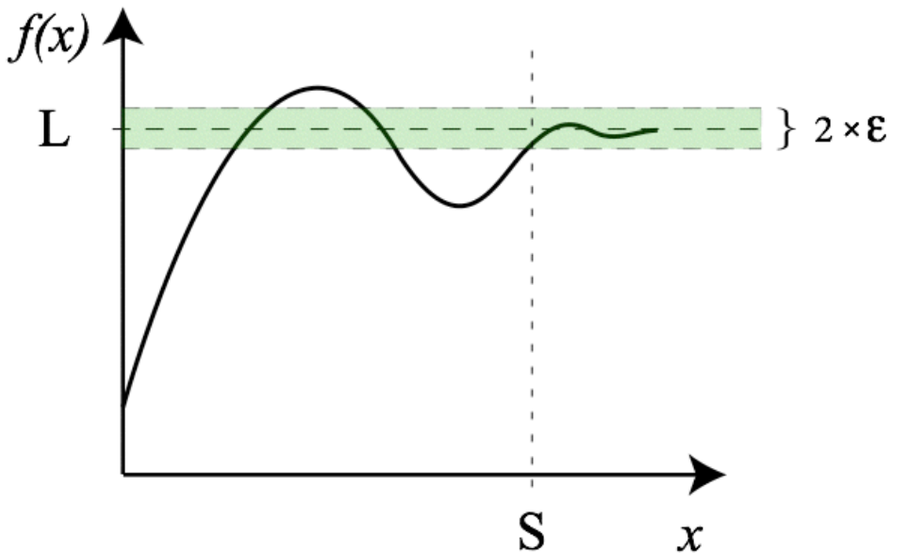


Figure 1.17
Definition of a Limit
Whenever a point x is within δ units of c, f(x) is within ε units of L.

Figure 1.18 Limit of a Function at Infinity



For an arbitrarily small ε, there always exists a large enough number N such that when x approaches N, |f(x)-L|< ε. Therefore, the limit of this function at infinity exists.

EXAMPLE

For a function graphed in *Figure 1.18*, for an arbitrarily small ε, there always exists a large enough number N such that when x approaches N, $|f(x)-L|< \varepsilon$. Therefore, the limit of this function at **infinity** exists.

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Continuity

A continuous function is a function for which, intuitively, "small" changes in the input result in "small" changes in the output.

KEY POINTS

- If a function is not continuous, it is said to be a "discontinuous function".
- The function f is continuous at some point c of its domain if the limit of f(x) as x approaches c through the domain of f exists and is equal to f(c).
- The function f is said to be continuous if it is continuous at every point of its domain.

A continuous function is a function for which, intuitively, "small" changes in the input result in "small" changes in the output. (*Figure 1.19*) Otherwise, a function is said to be a "discontinuous function". A continuous function with a continuous inverse function is called

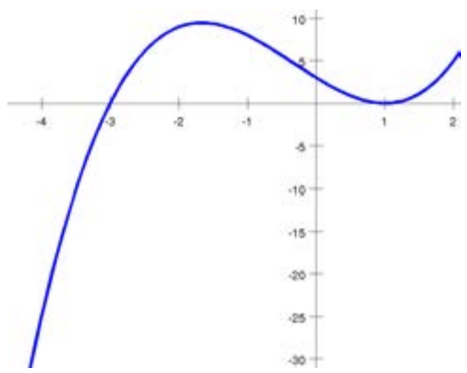

Figure 1.19
Continuity
The graph of a cubic function has no jumps or holes. The function is continuous.

"**bicontinuous**". Continuity of functions is one of the core concepts of **topology**.

The function f is continuous at some point c of its domain if the limit of f(x) as x approaches c through the domain of f exists and is equal to f(c). In mathematical notation, this is written as $\lim_{x \to c} f(x) = f(c)$.

In detail this means three conditions: first, f has to be defined at c. Second, the limit on the left hand side of that equation has to exist. Third, the value of this limit must equal f(c).

The function f is said to be continuous if it is continuous at every point of its domain. If the point c in the domain of f is not a limit point of the domain, then this condition is vacuously true, since x cannot approach c through values not equal c.

EXAMPLE

Consider the function h(t), which describes the height of a growing flower at time t. This function is continuous. In fact, a dictum of classical physics states that in nature everything is continuous. By contrast, if M(t) denotes the amount of money in a bank account at time t, then the function jumps whenever money is deposited or withdrawn, so the function M(t) is discontinuous.

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Finding Limits Algebraically

For a real-valued function expressed in terms of other functions, limit values may be computed via algebraic operations.

KEY POINTS

- Algebraic limit theorem states that

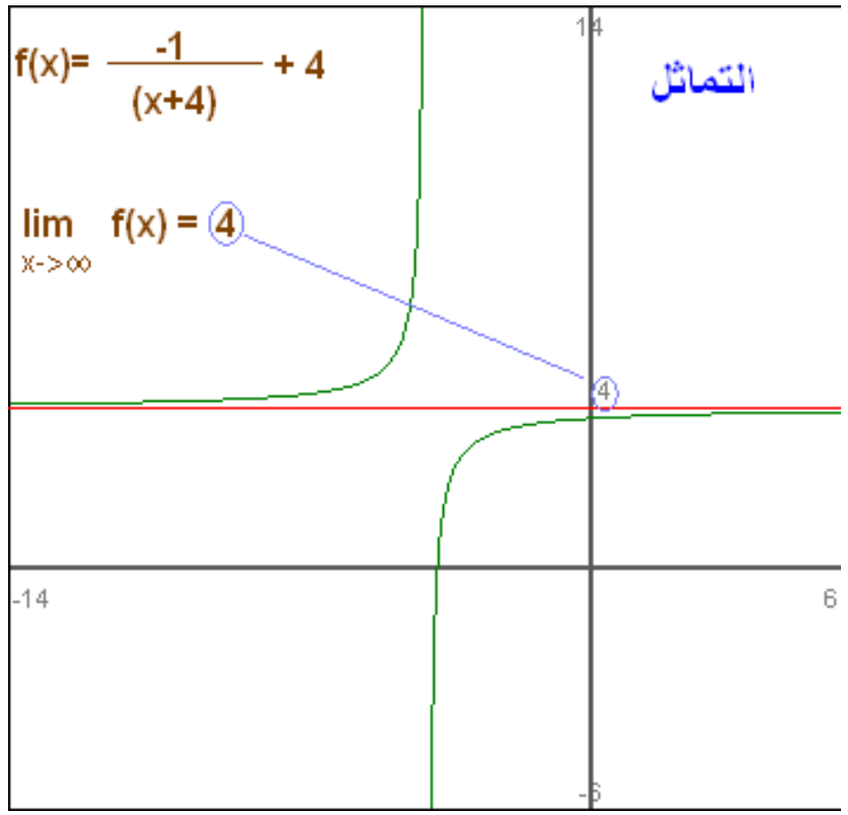
$$\lim_{x\to p} (f(x)+g(x)) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)-g(x)) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)\cdot g(x)) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)/g(x)) = \lim_{x\to p} f(x) / \lim_{x\to p} g(x)$$

- In each case above, when the limits on the right do not exist, nonetheless the limit on the left, called an indeterminate form, may still exist.
- When algebraic limit theorem doesn't yield a limit value, corresponding limits might often be determined with L'Hôpital's rule or the Squeeze theorem.

If f is a real-valued (or complex-valued) function, then taking the limit is compatible with the algebraic operations, provided the **limits** on the right sides of the equations below exist (the last identity only holds if the denominator is non-zero). This fact is often called the **algebraic** limit theorem (*Figure 1.20*):

$$\lim_{x\to p} (f(x)+g(x)) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)-g(x)) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)\cdot g(x)) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$$
$$\lim_{x\to p} (f(x)/g(x)) = \lim_{x\to p} f(x) / \lim_{x\to p} g(x)$$

Figure 1.20 Finding a Limit



The limit of f(x)=-1/(x+4) + 4 as x goes to infinity can be segmented down into two parts; limit of -1/(x+4) and limit of 4. The former is 0, while the latter is obviously 4. Therefore, the limit of f(x) as x goes to infinity is 4.

In each case above, when the limits on the right do not exist, or, in the last case, when the limits in both the numerator and the denominator are zero, nonetheless the limit on the left, called an

indeterminate form, may still exist—this depends on the functions f and g. These rules are also valid for one-sided limits, for the case p = ±∞, and also for infinite limits using the rules

q + ∞ = ∞ for q ≠ −∞

q × ∞ = ∞ if q > 0

q × ∞ = −∞ if q < 0

q/∞ = 0 if q ≠ ± ∞

Note that there is no general rule for the case q/0; it all depends on the way 0 is approached. Indeterminate forms—for instance, 0/0, 0×∞, ∞−∞, and ∞/∞—are also not covered by these rules, but the corresponding limits can often be determined with L'Hôpital's rule or the Squeeze theorem. We will study these rules in the following atoms.

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Trigonometric Limits

There are several limits of special interest involving trigonometric functions.

KEY POINTS

- The limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ is the most important relation involving limits of trigonometric functions.
- By substitute t=1/x in the first relation, we get $\lim_{x\to\infty} x \sin\left(\frac{c}{x}\right) = c.$

There are several limits of special interest involving **trigonometric functions**.

Limits of Special Interest

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$\lim_{x\to\infty} x \sin\left(\frac{c}{x}\right) = c$$

1. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

This limit can be proven with the squeeze theorem. For 0 < x < π/2: $\sin x < x < \tan x$. Dividing everything by sin(x) yields

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Taking the limit of the right hand side $\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$, the squeezing theorem tells us that $\lim_{x\to 0} \frac{x}{\sin x} = 1$. Equivalently, $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (*Figure 1.21*).

2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

This equation can be proven with the first limit and the following identity: $1 - \cos^2 x = \sin^2 x$. Starting with $\frac{1-\cos x}{x}$, multiplying numerator and denominator by (1 + cos x) yields

$$\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \frac{(1-\cos^2 x)}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x}\frac{\sin x}{1+\cos x}.$$

Using the algebraic limit theorem,

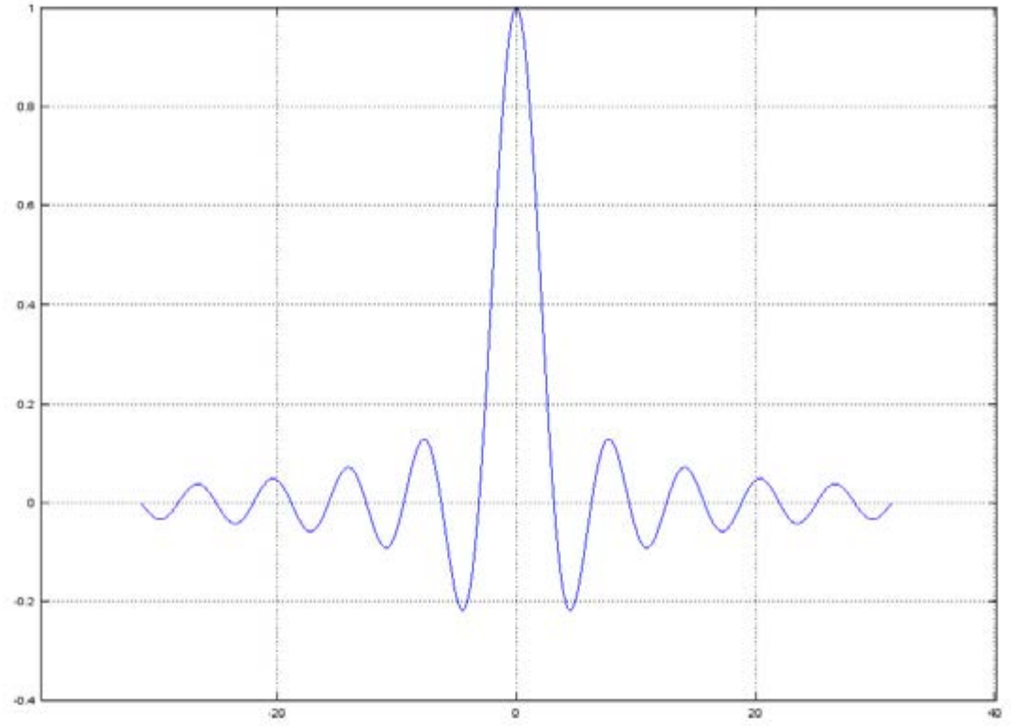
$$\lim_{x\to 0}\left(\frac{\sin x}{x}\frac{\sin x}{1+\cos x}\right) = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{2}\right) = 0.$$

Hence, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

3. $\lim_{x\to\infty} x \sin\left(\frac{c}{x}\right) = c$

This relation can be proven by substituting t=1/x and using the first relation we derived: $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Figure 1.21 Plot of f(x)=sin(x)/x.



The function is continuous at x=0, and f(x=0) is given as 1.

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Intermediate Value Theorem

For a real-valued continuous function f on the interval [a,b] and a number u between f(a) and f(b), there is a $c \in [a,b]$ such that f(c) = u.

KEY POINTS

- The intermediate value theorem states that for each value between the least upper bound and greatest lower bound of the image of a continuous function there is at least one point in its domain that the function maps to that value.
- The theorem is frequently stated in the following equivalent form: Suppose that $f : [a, b] \to R$ is continuous and that u is a real number satisfying $f(a) < u < f(b)$ or $f(a) > u > f(b)$. Then for some $c \in [a, b]$, $f(c) = u$.
- The theorem depends on (and is actually equivalent to) the completeness of the real numbers.

The intermediate value theorem states that for each value between the least upper bound and greatest lower bound of the image of a **continuous function** there is at least one point in its domain that the function maps to that value.

Version I: The intermediate value theorem states the following: If f is a real-valued continuous function on the interval [a, b], and u is a

number between f(a) and f(b), then there is a c ∈ [a, b] such that f(c) = u (*Figure 1.22*).

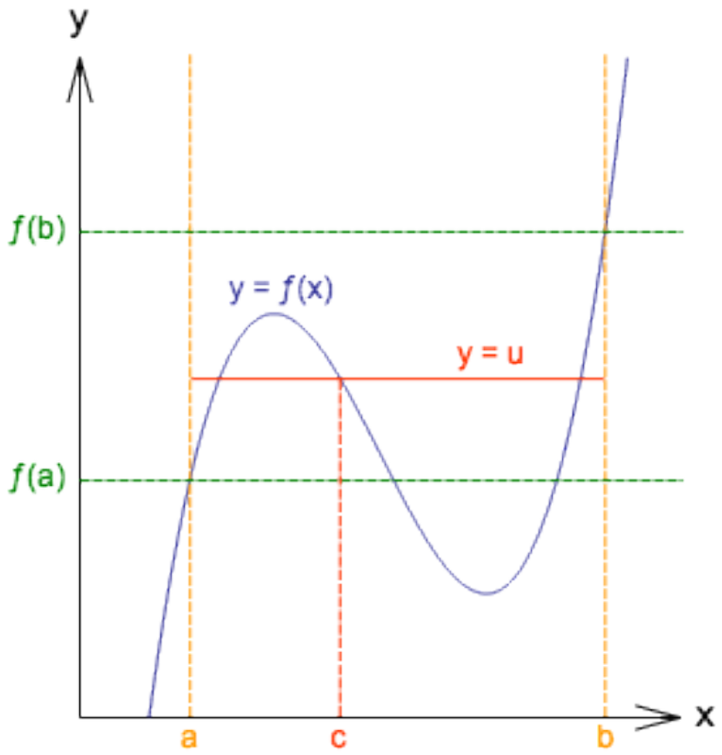

Figure 1.22
Intermedia Value Theorem
The intermediate value theorem states that if a function f is a real-valued continuous function on the interval [a, b], and u is a number between f(a) and f(b), then there is a c ∈ [a, b] such that f(c) = u.

Version II: Suppose that I is an interval [a, b] in the **real numbers** R and that f : I → R is a continuous function. Then the image set f(I) is also an interval, and either it contains [f(a), f(b)], or it contains [f(b), f(a)]; that is, f(I) ⊇ [f(a), f(b)], or f(I) ⊇ [f(b), f(a)].

It is frequently stated in the following equivalent form: Suppose that f : [a, b] → R is continuous and that u is a real number satisfying f(a) < u < f(b) or f(a) > u > f(b). Then for some c ∈ [a, b], f(c) = u.

This captures an intuitive property of continuous functions: given f continuous on [1, 2], if f(1) = 3 and f(2) = 5, then f must take the value 4 somewhere between 1 and 2. It represents the idea that the graph of a continuous function on a closed interval can be drawn without lifting your pencil from the paper.

The theorem depends on (and is actually equivalent to) the **completeness of the real numbers**. It is false for the rational numbers Q. For example, the function f(x) = x^2 − 2 for x ∈ Q satisfies f(0) = −2 and f(2) = 2. However there is no rational number x such that f(x) = 0, because √2 is irrational.

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Infinite Limits

Limits involving infinity can be formally defined using a slight variation of the (ε, δ)-definition.

KEY POINTS

- For f(x) a real function, the limit of f as x approaches infinity is L, means that for all $\varepsilon > 0$, there exists c such that whenever x > c $|f(x) - L| < \varepsilon$.
- For a rational function f(x) of the form p(x)/q(x), there are three basic rules for evaluating limits at infinity, where p(x) and q(x) are polynomials.
- If the limit at infinity exists, it represents a horizontal asymptote at y = L.

Limits involving infinity can be formally defined using a slight variation of the **(ε, δ)-definition**. For f(x) a real function, the limit of f as x approaches infinity is L, denoted $\lim_{x\to\infty} f(x) = L$, means that for all $\varepsilon > 0$, there exists c such that $|f(x) - L| < \varepsilon$ whenever x > c. Or, symbolically: $\forall\varepsilon > 0\ \exists c\ \forall x < c :\ |f(x) - L| < \varepsilon$ (*Figure 1.23*).

Similarly, the limit of f as x approaches negative infinity is L, denoted $\lim_{x\to-\infty} f(x) = L$, means that for all $\varepsilon > 0$ there exists c such that $|f(x) - L| < \varepsilon$ whenever x < c.

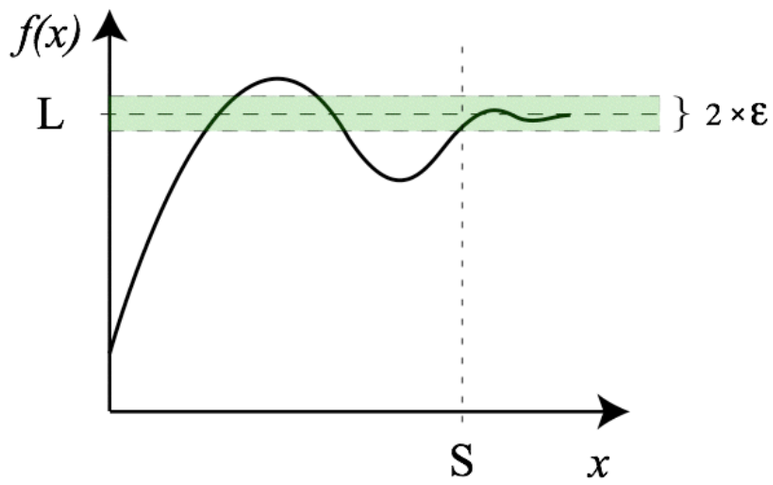

Figure 1.23 Infinite Limit

For any arbitrarily small ε, there exists a large enough N such that when x>N, |f(x)-2|<ε. Therefore, the limit of this function at infinity exists.

For a rational function f(x) of the form p(x)/q(x), there are three basic rules for evaluating limits at infinity (p(x) and q(x) are polynomials):

1. If the degree of p is greater than the degree of q, then the limit is positive or negative infinity depending on the signs of the leading coefficients;
2. If the degree of p and q are equal, the limit is the leading coefficient of p divided by the leading coefficient of q;
3. If the degree of p is less than the degree of q, the limit is 0.

If the limit at infinity exists, it represents a horizontal **asymptote** at y = L. Polynomials do not have horizontal asymptotes; such asymptotes may however occur with rational functions.

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Chapter 2

Derivatives and Integrals

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Section 1

Derivatives

The Derivative and Tangent Line Problem

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The Derivative and Tangent Line Problem

The use of differentiation makes it possible to solve the tangent line problem by finding the slope f'(a).

KEY POINTS

- The tangent line t (or simply the tangent) to a plane curve at a given point is the straight line that "just touches" the curve at that point.
- A straight line is said to be a tangent of a curve y = f(x) at a point x = c on the curve if the line passes through the point (c, f(c)) on the curve and has slope f'(c) where f' is the derivative of f.
- Using derivatives, the equation of the tangent line can be stated as follows: $y = f(a) + f(a)'(x - a)$

The **tangent** line t (or simply the tangent) to a plane curve at a given point is the straight line that "just touches" the curve at that point. Informally, it is a line through a pair of infinitely close points on the curve. More precisely, a straight line is said to be a tangent of a curve y = f(x) at a point x = c on the curve if the line passes through the point (c, f(c)) on the curve and has slope f'(c) where f' is the derivative of f (*Figure 2.1*).

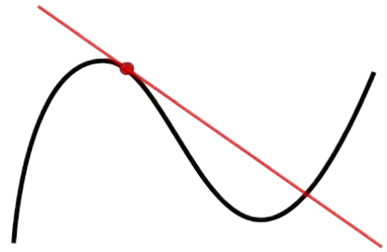

Figure 2.1 Tangent to a Curve

The line shows the tangent to the curve at the point represented by the dot. It barely touches the curve and shows the rate of change slope at the point.

Suppose that a curve is given as the graph of a function, y = f(x). To find the tangent line at the point p = (a, f(a)), consider another nearby point q = (a + h, f(a + h)) on the curve. The slope of the **secant** line passing through p and q is equal to the difference quotient

$$\frac{f(a+h)-f(a)}{h}.$$

As the point q approaches p, which corresponds to making h smaller and smaller, the difference quotient should approach a certain limiting value k, which is the slope of the tangent line at the point p. If k is known, the equation of the tangent line can be found in the point-slope form:

$$y - f(a) = k(x - a).$$

Suppose that the graph does not have a break or a sharp edge at p and it is neither plumb nor too wiggly near p. Then there is a unique value of k such that, as h approaches 0, the difference quotient gets closer and closer to k, and the distance between them becomes

negligible compared with the size of h, if h is small enough. This leads to the definition of the slope of the tangent line to the graph as the limit of the difference quotients for the function f. This limit is the derivative of the function f at x = a, denoted f ′(a). Using derivatives, the equation of the tangent line can be stated as follows:

$y = f(a) + f(a)^{'}(x - a).$

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Derivatives and Rates of Change

Differentiation is a way to calculate the rate of change of one variable with respect to another.

KEY POINTS

- Historically, the primary motivation for the study of differentiation was the tangent line problem: for a given curve: find the slope of the straight line that is tangent to the curve at a given point.
- If y is a linear function of x, then: $m = \frac{change\ in\ y}{change\ in\ x} = \frac{\triangle y}{\triangle x}$].
- The derivative measures the slope of a graph at each point.

Historically, the primary motivation for the study of differentiation was the tangent line problem: for a given curve: find the **slope** of the straight line that is tangent to the curve at a given point. The word tangent comes from the Latin word tangens, which means touching. Thus, to solve the tangent line problem, we need to find the slope of a line that is "touching" a given curve at a given point, or, in modern language, that has the same slope. But what exactly do we mean by "slope" for a curve?

The simplest case is when y is a linear function of x, meaning that the graph of y divided by x is a straight line. In this case, y = f(x) = m x + b, for real numbers m and b, and the slope m is given by

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{\triangle y}{\triangle x}$$

where the symbol Δ (the uppercase form of the Greek letter Delta) is an abbreviation for "change in." This formula is true because y + Δy = f(x+ Δx) = m (x + Δx) + b = m x + b + m Δx = y + mΔx.It follows that Δy = m Δx (*Figure 2.1*).

This gives an exact value for the slope of a straight line. If the function f is not linear (i.e. its graph is not a straight line), however, then the change in y divided by the change in x varies: differentiation is a method to find an exact value for this rate of change at any given value of x. In other words, differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x. This rate of change is called the derivative of y with respect to x. In more precise language, the dependence of y upon x means that y is a function of x. This functional relationship is often denoted y = f(x), where f denotes the function. If x and y are real numbers, and if the graph of y is plotted against x, the derivative measures the slope of this graph at each point.

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The Derivative as a Function

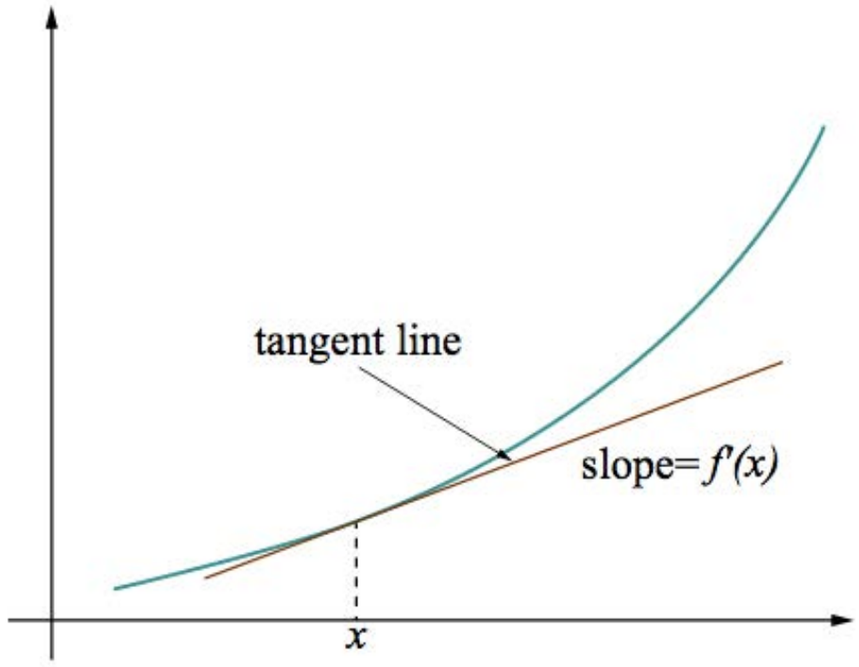
If every point of a function has a derivative, there is a derivative function sending the point a to the derivative of f at x = a: f'(a).

KEY POINTS

- The derivative is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions.
- The function whose value at x = a equals f′(a) whenever f′(a) is defined and elsewhere is undefined is also called the derivative of f.
- By the definition of the derivative function, D(f)(a) = f′(a), where D is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions.

Let f be a function that has a derivative at every point a in the **domain** of f. Because every point a has a derivative, there is a function that sends the point a to the derivative of f at a. This function is written f′(x)and is called the derivative function or the derivative of f. The derivative of f collects all the derivatives of f at all the points in the domain of f. Visually, derivative of a function f at x=a represents the slope of the curve at the point x=a (*Figure 2.2*).

Figure 2.2 The Derivative of a Function



The derivative of the function f(x) is equivalent to the slope of the curve at the point of evaluation.

Sometimes f has a derivative at most, but not all, points of its domain. The function whose value at a equals f′(a) whenever f′(a) is defined and elsewhere is undefined is also called the derivative of f. It is still a function, but its domain is strictly smaller than the domain of f (*Figure 2.3*).

Using this idea, differentiation becomes a function of functions: The derivative is an operator whose domain is the set of all functions

that have derivatives at every point of their domain and whose **range** is a set of functions. If we denote this operator by D, then D(f) is the function f′(x). Since D(f) is a function, it can be evaluated at a point a. By the definition of the derivative function, D(f)(a) = f′(a).

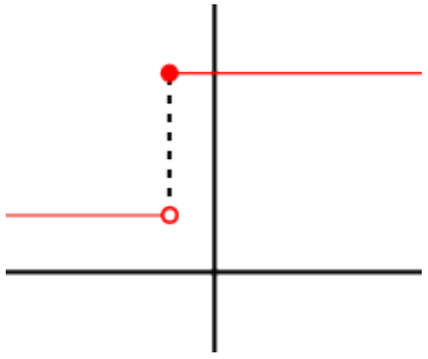


Figure 2.3 Function with Discontinuity

There is no associated derivation function since there is no derivative at the discontinuity.

For comparison, consider the doubling function f(x) = 2x; f is a real-valued function of a real number, meaning that it takes numbers as inputs and has numbers as outputs:

$1 \to 2$

$2 \to 4$

$3 \to 6.$

The operator D, however, is not defined on individual numbers. It is only defined on functions:

$D(x \to 1) = (x \to 0)$

$D(x \to x) = (x \to 1)$

$D(x \to x^2) = (x \to 2x).$

Because the output of D is a function, the output of D can be evaluated at a point. For instance, when D is applied to the squaring function,

$x \to x^2,$

D outputs the doubling function,

$x \to 2x,$

which is f(x). This output function can then be evaluated to get f(1) = 2, f(2) = 4, and so on.

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Differentiation Rules

The rules of differentiation can simplify derivatives by eliminating the need for complicated limit calculations.

KEY POINTS

- Differentiation by polynomial expansion can be very complicated and prone to errors.
- Constant rule: if f(x) is a constant, then its derivative f'(x) = 0.
- Chain Rule: If f(x) = h(g(x)), then f'(x) = h'(g(x)) g'(x)
- Product Rule: $(fg)' = f'g + g'f$
- Quotient Rule: $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$

When we wish to differentiate complicated expressions, a possible way to differentiate the expression is to expand it and get a **polynomial**, and then differentiate that polynomial. This method becomes very complicated and is particularly error prone when doing calculations by hand. In many cases, complicated limit calculations by direct application of Newton's **difference quotient** can be avoided by using differentiation rules. Some of the most basic rules are the following.

Constant rule: if f(x) is a constant, then its derivative f'(x) = 0, since the rate of chance of a constant is obviously zero.

Sum Rule: for all functions f and g and all real numbers α and β, then $(\alpha f + \beta g)' = \alpha f' + \beta g'$.

Product Rule: $(fg)' = f'g + g'f$ for all functions f and g. By extension, this means that the derivative of a constant times a function is the constant times the derivative of the function: $\frac{d}{dr}\pi r^2 = 2\pi r$.

Quotient Rule: $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ for all functions f and g at all inputs where g is not equal to 0.

Figure 2.4 is an example of modeling that uses the product rule.

Figure 2.4 Model Rockets The flight of model rockets can be modeled using the product rule.

Chain Rule: If f(x) = h(g(x)), then f'(x) = h'(g(x)) g'(x).

For example, consider $f(x) = x^4 + sin(x^2) - ln(x)e^x + 7$, then

$$f'(x) = 4x^{(4-1)} + \frac{d(x^2)}{dx}cos(x^2) - \frac{d(ln\ x)}{dx}e^x - ln\ x\frac{d(e^x)}{dx} + 0$$

$= 4x^3 + 2xcos(x^2) - \frac{1}{x}e^x - ln(x)e^x$. Here the second term was computed using the chain rule and third using the product rule. The known derivatives of the elementary functions $x^2, x^4, sin(x), ln(x)\ and\ e^x$, as well as the constant 7, were also used.

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Derivatives of Trigonometric Functions

Derivatives of trigonometric functions can be found using the standard derivative formula.

KEY POINTS

- The derivative of the sine function is the cosine function.
- The derivative of the cosine function is the negative of the sine function.
- The derivative of the tangent function is the squared secant function.

The trigonometric functions (also called the circular functions) are functions of an angle. They are used to relate the angles of a triangle to the lengths of the sides of a triangle. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.

The most familiar trigonometric functions are the sine, cosine, and tangent. In the context of the standard unit circle with radius 1, where a triangle is formed by a ray originating at the origin and making some angle with the x-axis, the sine of the angle gives the length of the y-component (rise) of the triangle, the cosine gives the

length of the x-component (run), and the tangent function gives the slope (y-component divided by the x-component).

With this in mind, we can use the definition of a derivative,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to calculate the derivatives of different trigonometric functions. For example, if $f(x) = sin\ x$, then

$$f'(x) = \lim_{h \to 0} \frac{sin\ (x+h) - sin\ (x)}{h}$$

$$= \lim_{h \to 0} \frac{cos(x)sin(h) + cos(h)sin(x) - sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{cos(x)sin(h) + (cos(h) - 1)sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{cos(x)sin(h)}{h} + \lim_{h \to 0} \frac{(cos(h) - 1)sin(x)}{h}$$

$$= cos(x)(1) + sin(x)(0)$$

$$= cos(x).$$

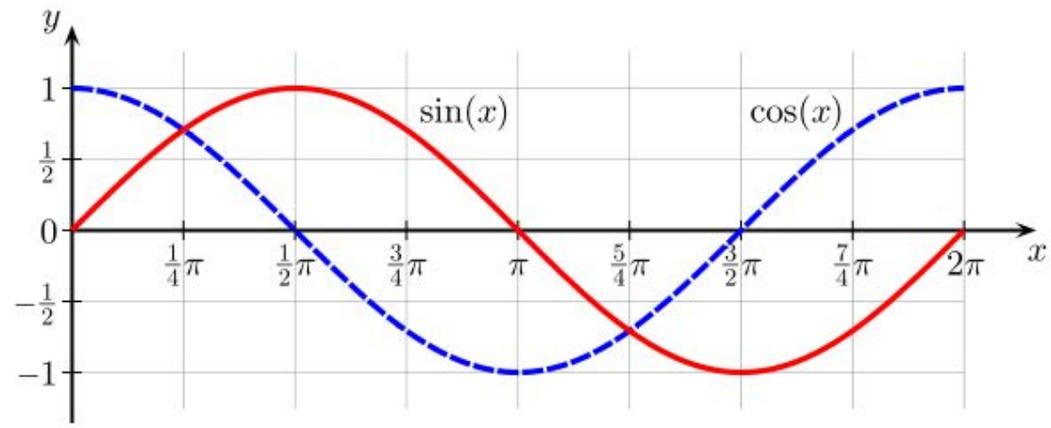
See *Figure 2.5* for graphical validation.

The same procedure can be applied to find other derivatives of trigonometric functions, the most common being

$$\frac{\mathrm{d}}{\mathrm{d}x} cos\ x = -\ sin\ x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} tan\ x = sec^2\ x.$$

Figure 2.5 Sine and Cosine Function



The derivative of the sine function is the cosine function, and the derivative of the cosine function is negative the sine function.

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The Chain Rule

The chain rule is a formula for computing the derivative of the composition of two or more functions.

KEY POINTS

- If f is a function and g is a function, then the chain rule expresses the derivative of the composite function f ∘ g in terms of the derivatives of f and g.
- The chain rule can be applied sequentially for as many functions as are nested inside one another.
- The chain rule for f ∘ g (x) is $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

The **chain rule** is a formula for computing the derivative of the composition of two or more functions. That is, if f is a function and g is a function, then the chain rule expresses the derivative of the **composite** function f ∘ g in terms of the derivatives of f and g. For example, the chain rule for f ∘ g (x) ≡ f [g (x)] is $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

The method is called the "chain rule" because it can be applied sequentially to as many functions as are nested inside one another. For example, if f is a function of g, which is in turn a function of h, which is in turn a function of x--that is, f(g(h(x)))-- then the derivative of f with respect to x is given by $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$.

The chain rule has broad applications in physics, chemistry, and engineering, as well as in the study of related rates in many disciplines. The chain rule can also be generalized to multiple variables in cases where the nested functions depend on more than one variable (*Figure 2.6*).

Figure 2.6 Skydiving

The path of a skydiver relies on many variables such as time and height. Use of the chain rule is needed for the complicated calculation.

Consider the function f(x) = (x² + 1)³. It follows from the chain rule that

$$f(x) = (x^2 + 1)^3$$

$$u(x) = x^2 + 1$$

$$f(x) = [u(x)]^3$$

Substituting f(u) and u(x), we get

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$= \frac{d}{du}u^3 \frac{d}{dx}(x^2 + 1)$$

$$= (3u^2)(2x)$$

$$= (3(x^2 + 1)^2)(2x)$$

$$= 6x(x^2 + 1)^2.$$

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Implicit Differentiation

Implicit differentiation makes use of the chain rule to differentiate implicitly defined functions.

KEY POINTS

- As y can be given as a function of x implicitly rather than explicitly, when we have an equation R(x, y) = 0, we may be able to solve it for y and then differentiate.
- An implicit function is a function that is defined implicitly by a relation between its argument and its value.
- The implicit function theorem states that if the left hand side of the equation R(x, y) = 0 is differentiable and satisfies some mild condition on its partial derivatives at some point (a, b) such that R(a, b) = 0, then it defines a function y = f(x) over some interval containing a.

An **implicit** function is a function that is defined implicitly by a relation between its argument and its value.

The implicit function theorem provides a condition under which a relation defines an implicit function. It states that if the left hand side of the equation R(x, y) = 0 is differentiable and satisfies some mild condition on its partial derivatives at some point (a, b) such that R(a, b) = 0, then it defines a function y = f(x) over some

interval containing a. Geometrically, the graph defined by R(x,y) = 0 will overlap locally with the graph of some equation y = f(x).

For most implicit functions, there is no formula which defines them explicitly. However, various numerical methods exist for computing approximately the value of y corresponding to any fixed value of x; this allows us to find an explicit approximation to the implicit function.

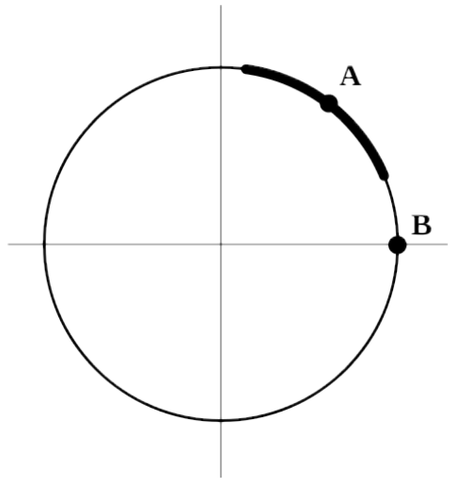

Figure 2.7
Path of a Point on a Circle
The path of a point on a circle can only be expressed as an implicit function.

Implicit differentiation makes use of the chain rule to differentiate implicitly defined functions.

As y can be given as a function of x implicitly rather than explicitly, when we have an equation R(x, y) = 0, we may be able to solve it for y and then differentiate. However, sometimes it is simpler to differentiate R(x, y) with respect to x and y and then solve for dy/dx.

For example, given an expression y + x + 5 = 0, one can differentiate to obtain $\frac{dy}{dx} + \frac{dx}{dx} + \frac{d}{dx}5 = \frac{dy}{x} + 1 = 0$. Solving for dy/dx gives $\frac{dy}{dx} = -1$ (*Figure 2.7*).

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Differentiation and Rates of Change in the Natural and Social Sciences

Differentiation, in essence calculating the rate of change, is important in all quantitative sciences.

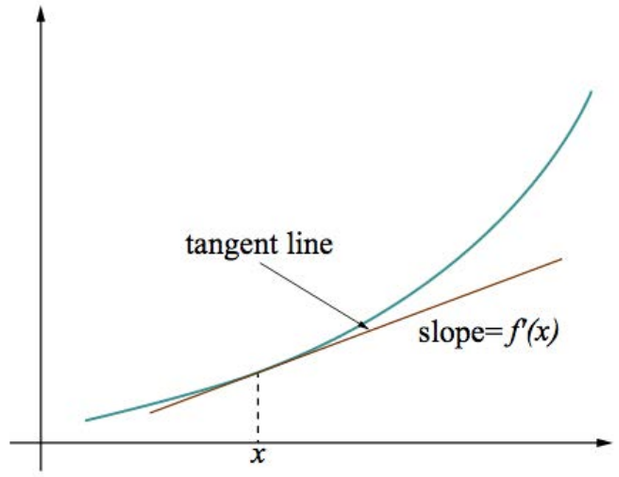
KEY POINTS

- Differentiation has applications to nearly all quantitative disciplines, whether it's natural or social science.
- Physical scientists use differentiation and rate of change to study the way a physical concept changes over time or distance.
- Social scientists use differentiation and rate to determine how people, goods, and processes change due to the change of an independent variable.

Given a function y=f(x), differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x. This rate of change is called the derivative of y with respect to x.

Rate of change is an important concept in many quantitative studies, and it is no surprise that differentiation (representing the rate of change) has applications to nearly all quantitative disciplines. For example, in physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of velocity with respect to time is acceleration.

Figure 2.8 Analytical Function



Any function that describes the rate of change of a quantity can be examined by differentiation, whether in physical or social sciences. Here, the slope of the tangent line is given by the derivative of the function representing the curve.

Newton's second law of motion states that the derivative of the **momentum** of a body equals the force applied to the body. The reaction rate of a chemical reaction is a derivative. In operations research, derivatives determine the most efficient ways to transport materials and design factories.

Derivatives are frequently used to find the maxima and minima of a function. Equations involving derivatives are called differential equations and are fundamental in describing natural phenomena. Derivatives and their generalizations appear in many fields of mathematics, such as complex analysis, functional analysis,

differential geometry, measure theory and abstract algebra (*Figure 2.8*).

Rates of change occur in all sciences and across all disciplines. Economists study the rate of change of **gross domestic product** and social scientists the rate in which populations vote in a specific area. Geologists study the rate of earth shift and the temperature gradient of rocks near a volcano. Accountants study the rate of change of production and supplies, and how any change can affect cost and profit. Urban engineers study the flow of traffic in order to design and build more efficient roads and freeways.

In every aspect of life in which something changes, differentiation and rates of change are an important aspect in understanding the world and finding ways to improve it.

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Related Rates

Related rates problems involve finding a rate by relating that quantity to other quantities whose rates of change are known.

KEY POINTS

- Because science and engineering often relate quantities to each other, the methods of related rates have broad applications in these fields.
- Because problems involve several variables, differentiation with respect to time or one of the other variables requires application of the chain rule.
- The process for solving related rates problems: Write out any relevant formulas and information, take the derivative of the primary equation with respect to time, solve for the desired variable, plug in known information and simplify.

One useful application of derivatives is as an aid in the calculation of related rates. What is a related rate? In differential calculus, related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. In most cases, the related rate that is being calculated is a derivative with respect to some value. We compute this derivative from a rate at which some other known quantity is

changing. Given the rate at which something is changing, we are asked to find the rate at which a value related to the rate we are given is changing.

The rate of change is usually with respect to time. Because science and engineering often relate quantities to each other, the methods of related rates have broad applications in these fields. Because problems involve several **variables**, differentiation with respect to time or one of the other variables requires application of the chain rule.

Process for solving related rates problems:

1. Write out any relevant formulas and information.

2. Take the derivative of the primary equation with respect to time.

3. Solve for the desired variable.

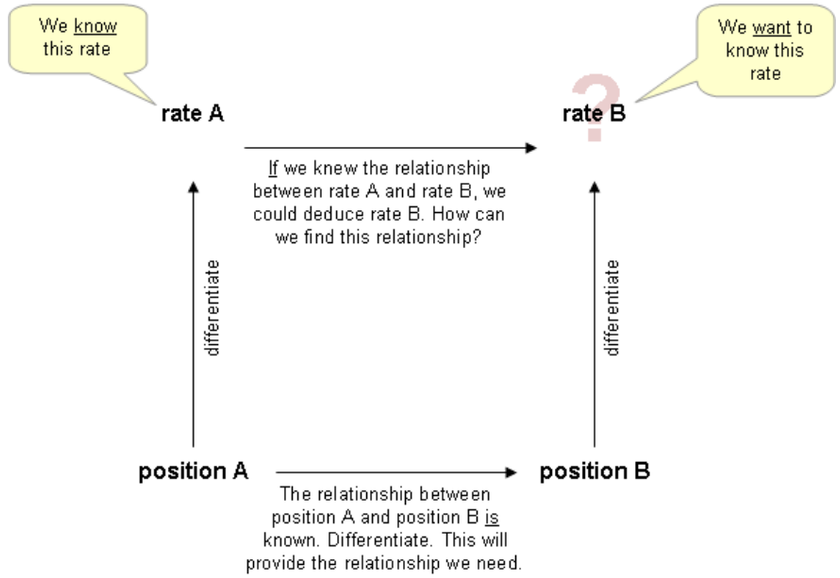
4. Plug in known information and simplify.

See *Figure 2.9* for a flowchart for solving related rates problems.

Suppose you are given the following situation:

A spherical hot air balloon is being filled with air. The volume is changing at a rate of 2 cubic feet per minute. How is the radius changing with respect to time when the radius is equal to 2 feet?

Figure 2.9 Flow Chart for Related Rate Problem Solving


We know this rate
We want to know this rate
rate A
rate B
?
If we knew the relationship between rate A and rate B, we could deduce rate B. How can we find this relationship?
differentiate
differentiate
position A
position B
The relationship between position A and position B is known. Differentiate. This will provide the relationship we need.

Related rate problems can be handled by taking a methodical approach.

The relevant formulas and pieces of information are the volume of the balloon, the rate of change of the volume, and the radius.

$$V = \frac{4}{3}\pi r^3$$

$$V' = 2$$

$$r = 2.$$

Take the derivative of both sides with respect to time.

$$V' = \frac{4}{3}(3)\pi r^2 r'$$

$$= 4\pi r^2 r'.$$

Solving for r,

$$r' = \frac{1}{4\pi r^2} V',$$

and plugging in the known information,

$$r' = \frac{1}{16\pi} 2 = \frac{1}{8\pi} ft/min.$$

EXAMPLE

Suppose you are given the following situation:

A spherical hot air balloon is being filled with air. The volume is changing at a rate of 2 cubic feet per minute. How is the radius changing with respect to time when the radius is equal to 2 feet?

The relevant formulas and pieces of information are the volume of the balloon, the rate of change of the volume, and the radius.

$$r' = \frac{1}{16\pi} 2 = \frac{1}{8\pi} ft/min$$

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Higher Derivatives

The derivative of a differentiated expression is called a higher-order derivative.

KEY POINTS

- The second derivative, or second order derivative, is the derivative of the derivative of a function.
- Because the derivative of function is defined as a function representing the slope of function, the double derivative is the function representing the slope of the first derivative function.
- If x(t) represents the position of an object at time t, then the higher-order derivatives of x have physical interpretations, such as velocity and acceleration.

The second **derivative**, or second order derivative, is the derivative of the derivative of a function. The derivative of the function may be denoted by f'(x), and its double (or "second") derivative is denoted by f''(x). This is read as "f double prime of x," or "the second derivative of f(x)." Because the derivative of a function is defined as a function representing the slope of function, the double derivative is the function representing the slope of the first derivative function.

Furthermore, the third derivative is the derivative of the derivative of the derivative of a function, which can be represented by f'''(x).

This is read as "f triple prime of x", or "the third derivative of f(x)". This can continue as long as the resulting derivative is itself differentiable, with the fourth derivative, the fifth derivative, and so on. Any derivative beyond the first derivative can be referred to as a higher order derivative.

If x(t) represents the position of an object at time t, then the higher-order derivatives of x have physical interpretations. The second derivative of x is the derivative of x′(t), the velocity, and by definition is the object's acceleration. The third derivative of x is defined to be the jerk, and the fourth derivative is defined to be the jounce. A function f need not have a derivative--for example, if it is not continuous. Similarly, even if f does have a derivative, it may not have a second derivative. See *Figure 2.10* for a graphical illustration of higher derivatives in physics.

An example of a function with higher-order derivatives is

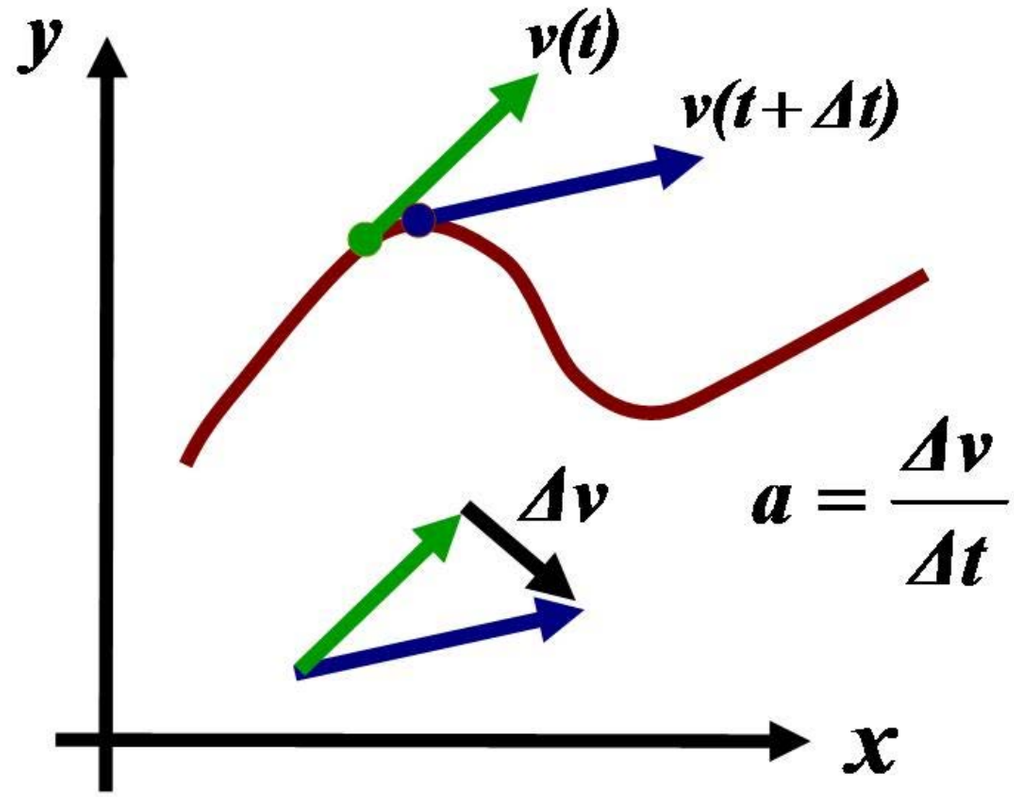
$f(x) = 5x^3 + 3x^2 - x + 4$, where the higher derivatives are found to be

$f'(x) = 15x^2 + 6x - 1$,

$f''(x) = 30x + 6$, and

$f'''(x) = 30$.

Figure 2.10 Acceleration



Acceleration is the time-rate of change of velocity, and the second-order rate of change of position.

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Section 2

Applications of Differentiation

Linear Approximation

Maximum and Minimum Values

The Mean Value Theorem, Rolle's Theorem, and Monotonicity

Derivatives and the Shape of the Graph

Horizontal Asymptotes and Limits at Infinity

Curve Sketching

Graphing on Computers and Calculators

Optimization

Newton's Method

Concavity and the Second Derivative

Differentials

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Linear Approximation

A linear approximation is an approximation of a general function using a linear function.

KEY POINTS

- By taking the derivative one may find the slope of a function.
- The values between two points can be approximated as lying on a straight line between those points, where the line is tangent to the function at one of the points.
- Linear approximation can be made arbitrarily accurate by decreasing the distance between the sample points.

In mathematics, a linear approximation is an approximation of a general function using a **linear** function (more precisely, an affine function).

They are widely used in the method of finite differences to produce first order methods for solving or approximating solutions to equations.

Given a twice continuously **differentiable** function f of one real variable, Taylor's theorem for the case n = 1 states that

$f(x) = f(a) + f'(a)(x - a) + R_2$ where R_2 is the remainder term.

The linear approximation is obtained by dropping the remainder:

$$f(x) = f(a) + f'(a)(x - a)$$

This is a good approximation for x when it is close enough to a; since a curve, when closely observed, will begin to resemble a straight line, as seen in *Figure 2.11*. Therefore, the expression on the right-hand side is just the equation for the tangent line to the graph of f at (a,f(a)). For this reason, this process is also called the tangent line approximation.

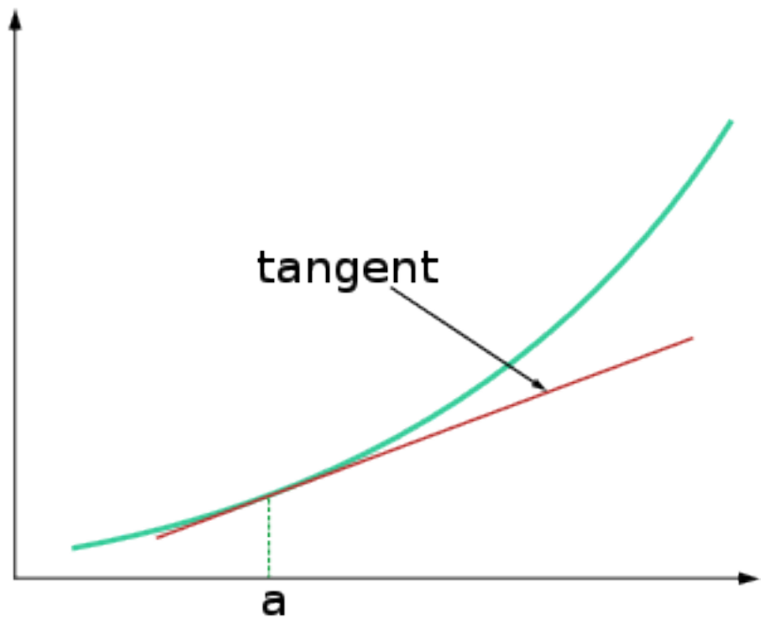

Figure 2.11 Linear Approximation
Tangent line at (a, f(a))

If f is concave down in the interval between x and a, the approximation will be an overestimate (since the derivative is decreasing in that interval). If f is concave up, the approximation will be an underestimate.

Since the line tangent to the graph is given by the derivative, differentiation is useful for finding the linear approximation. If one were to take an infinitesimally small step size for a, the linear approximation would exactly match the function.

Linear approximations for vector functions of a vector variable are obtained in the same way, with the derivative at a point replaced by the Jacobian matrix. For example, given a differentiable function with real values, one can approximate for close to by the formula

$$f(x, y)\ f(a, b) + \frac{df}{dx}(a, b)(x - a) + \frac{df}{dy}(a, b)(y - b).$$

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Maximum and Minimum Values

Maxima and minima are critical points on graphs and can be found by the first derivative and the second derivative.

KEY POINTS

- The critical point of a function is where the first derivative is 0.
- A critical point often indicates a maximum or a minimum.
- If the second derivative at a critical point is positive then it is a minimum and if it is positive then it is a minimum.

In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point either within a given neighborhood (local or relative extremum) or on the function domain in its entirety (global or absolute extremum, *Figure 2.12*).

A real-valued function f defined on a real line is said to have a local (or relative) maximum point at the point $x*$, if there exists some $\varepsilon > 0$ such that $f(x*) \geq f(x)$ when $|x - x*| < \varepsilon$. The value of the function

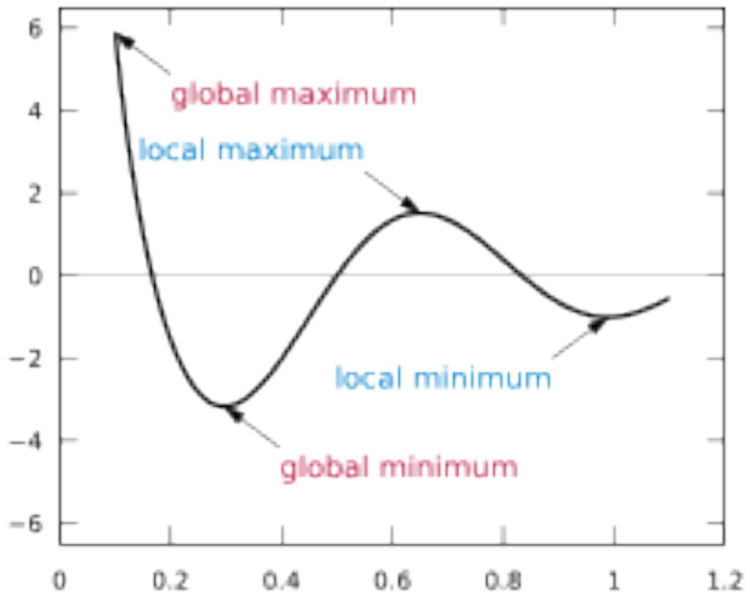

Figure 2.12
Maxima and Minima

Local and global maxima and minima for $\cos(3\pi x)/x$, $0.1 \leq x \leq 1.1$

at this point is called maximum of the function. Similarly, a function has a local minimum point at $x*$, if $f(x*) \leq f(x)$ when $|x - x*| < \varepsilon$. The value of the function at this point is called minimum of the function.A function has a global (or absolute) maximum point at $x*$ if $f(x*) \geq f(x)$ for all x. Similarly, a function has a global (or absolute) minimum point at $x*$ if $f(x*) \leq f(x)$ for all x. The global maximum and global minimum points are also known as the arg max and arg min: the argument (input) at which the maximum (respectively, minimum) occurs.

Finding global maxima and minima is the goal of mathematical optimization. If a function is continuous on a closed interval, then by the extreme value theorem global maxima and minima exist.

Furthermore, a global maximum (or minimum) either must be a local maximum (or minimum) in the interior of the domain, or must lie on the boundary of the domain. So a method of finding a global maximum (or minimum) is to look at all the local maxima (or minima) in the interior, and also look at the maxima (or minima) of the points on the boundary; and take the biggest (or smallest) one.Local extrema can be found by Fermat's theorem, which states that they must occur at critical points. One can distinguish whether a critical point is a local maximum or local minimum by using the first derivative test or second derivative test.

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The Mean Value Theorem, Rolle's Theorem, and Monotonicity

The MVT states that for a function continuous on an interval, the mean value of the function on the interval is a value of the function.

KEY POINTS

- In calculus, the mean value theorem states, roughly: given a planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints.
- More precisely, if a function f is continuous on the closed interval [a, b], where a < b, and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- Rolle's Theorem states that if a real-valued function f is continuous on a closed interval [a, b], differentiable on the open interval (a, b), and f(a) = f(b), then there exists a c in the open interval (a, b) such that f'(c)=0.

In calculus, the **mean** value theorem states, roughly: given a planar arc between two endpoints, there is at least one point at which the **tangent** to the arc is parallel to the **secant** through its endpoints (*Figure 2.13*).

The theorem is used to prove global statements about a function on an interval starting from local hypotheses about derivatives at points of the interval. More precisely, if a function f is continuous on the closed interval [a, b], where a < b, and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

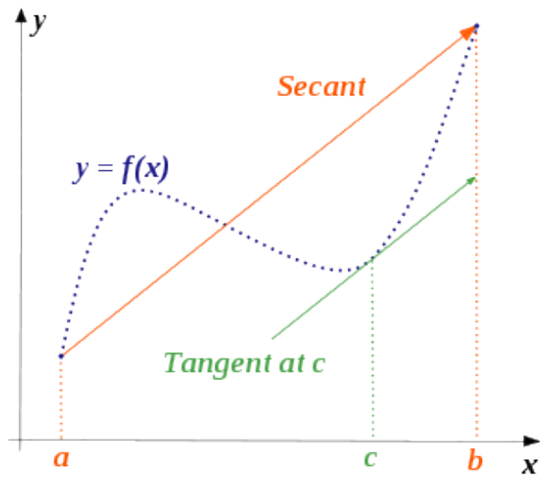
This theorem can be understood intuitively by applying it to motion: If a car travels one hundred miles in one hour, then its average speed during that time was 100 miles per hour. To get at that average speed, the car either has to go at a constant 100 miles per hour during that whole time, or, if it goes slower at one moment, it has to go faster at another moment as well (and vice versa), in order to still end up with an average of 100 miles per hour. Therefore, the Mean Value Theorem tells us that at some point during the journey, the car must have been traveling at exactly 100 miles per hour; that is, it was traveling at its average speed.

The mean value theorem follows from the more specific statement of Rolle's theorem, and can be used to prove the more general

statement of Taylor's theorem (with Lagrange form of the remainder term).

Rolle's Theorem states that if a real-valued function f is continuous on a closed interval [a, b], differentiable on the open interval (a, b), and f(a) = f(b), then there exists a c in the open interval (a, b) such that $f'(c) = 0$.

Figure 2.13 Mean Value Theorem



For any function that is continuous on [a, b] and differentiable on (a, b) there exists some c in the interval (a, b) such that the secant joining the endpoints of the interval [a, b] is parallel to the tangent at c.

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Derivatives and the Shape of the Graph

The shape of a graph may be found by taking derivatives to tell you the slope and concavity.

KEY POINTS

- The derivative of a function is the the function that defines the slope of the graph at each point.
- The second derivative of the graph tells you the concavity of the graph at a point.
- Inflection points are where the second derivative is 0 and are points where the concavity changes.

Differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x. This rate of change is called the derivative of y with respect to x. In more precise language, the dependence of y upon x means that y is a function of x. This functional relationship is often denoted y = f(x), where f denotes the function.

If x and y are real numbers, and if the graph of y is plotted against x, the derivative measures the slope of this graph at each point (*Figure 2.14*).

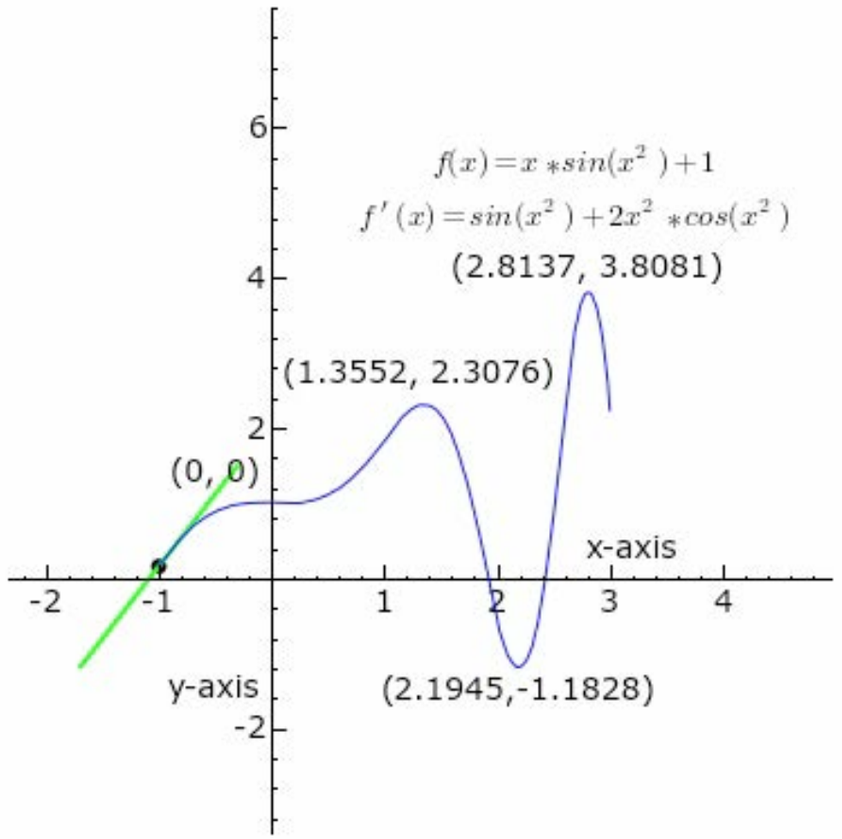
The simplest case is when y is a linear function of x, meaning that the graph of y divided by x is a straight line. In this case, y = f(x) = m x + b, for real numbers m and b, and the slope m is given by $\frac{\Delta y}{\Delta x}$ where the symbol Δ (the uppercase form of the Greek letter Delta) is an abbreviation for "change in."

This formula is true becausey + Δy = f(x+ Δx) = m (x + Δx) + b = m x + b + m Δx = y + mΔx. It follows that Δy = m Δx. This gives an exact value for the slope of a straight line. If the function f is not linear (i.e. its graph is not a straight line), however, then the change in y divided by the change in x varies: differentiation is a method to find an exact value for this rate of change at any given value of x.

Inflection Point

A point where the second derivative of a function changes sign is called an inflection point. At an inflection point, the second derivative may be zero, as in the case of the inflection point x=0 of the function y=x3, or it may fail to exist, as in the case of the inflection point x=0 of the function y=x1/3. At an inflection point, a function switches from being a **convex** function to being a **concave** function or vice versa.

Figure 2.14 Derivative



At each point, the derivative of is the slope of a line that is tangent to the curve. The line is always tangent to the blue curve; its slope is the derivative. Note derivative is positive where a green line appears, negative where a red line appears, and zero where a black line appears.

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Horizontal Asymptotes and Limits at Infinity

The asymptotes are computed using limits and are classified into horizontal, vertical and oblique depending on the orientation.

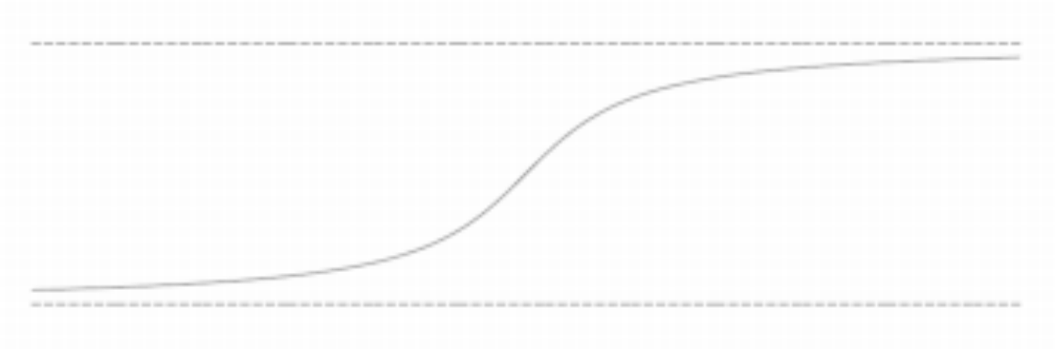
KEY POINTS

- Horizontal asymptotes are horizontal lines that the graph of the function approaches as x tends to +∞ or −∞.
- Vertical asymptotes are vertical lines (perpendicular to the x-axis) near which the function grows without bound.
- Oblique asymptotes are diagonal lines so that the difference between the curve and the line approaches 0 as x tends to +∞ or −∞.

Differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x. This rate of change is called the derivative of y with respect to x. In more precise language, the dependence of y upon x means that y is a function of x. This functional relationship is often denoted y = f(x), where f denotes the function.

If x and y are real numbers, and if the graph of y is plotted against x, the derivative measures the slope of this graph at each point (*Figure 2.15*).

Figure 2.15 Horizontal asymptote



The graph of a function can have two horizontal asymptotes. An example of such a function would be y=arctan(x).

The simplest case is when y is a linear function of x, meaning that the graph of y divided by x is a straight line. In this case, y = f(x) = m x + b, for real numbers m and b, and the slope m is given by $\frac{\Delta y}{\Delta x}$ where the symbol Î” (the uppercase form of the Greek letter Delta) is an abbreviation for "change in."

This formula is true because y + Î”y = f(x+ Î”x) = m (x + Î”x) + b = m x + b + m Î”x = y + mÎ”x. It follows that Î”y = m Î”x. This gives an exact value for the slope of a straight line. If the function f is not linear (i.e. its graph is not a straight line), however, then the change in y divided by the change in x varies: differentiation is a method to find an exact value for this rate of change at any given value of x.

Inflection Point

A point where the second derivative of a function changes sign is called an inflection point. At an inflection point, the second derivative may be zero, as in the case of the inflection point x=0 of the function y=x3, or it may fail to exist, as in the case of the inflection point x=0 of the function y=x1/3. At an inflection point, a function switches from being a convex function to being a concave function or vice versa.

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Curve Sketching

Curve sketching is used to produce a rough idea of overall shape of a curve given its equation without computing a detailed plot.

KEY POINTS

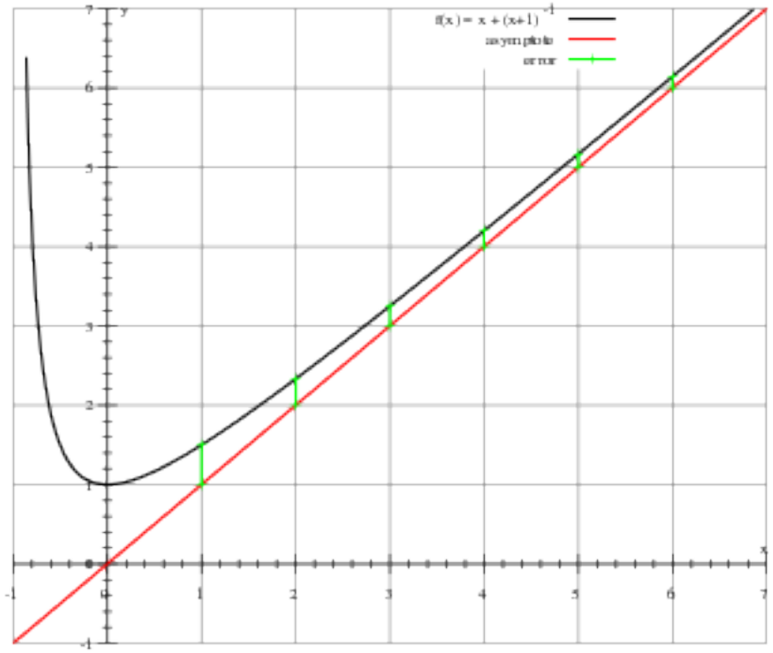
- Determine the x and y intercepts of the curve.
- Determine the symmetry of the curve.
- Determine any bounds on the values of x and y.
- Determine the asymptotes of the curve.

In geometry, curve sketching (or curve tracing) includes techniques that can be used to produce a rough idea of overall shape of a plane curve given its equation without computing the large numbers of points required for a detailed plot. It is an application of the theory of curves to find their main features.

The following are usually easy to carry out and give important clues as to the shape of a curve:

Determine the x and y intercepts of the curve. The x intercepts are found by setting y equal to 0 in the equation of the curve and solving for x. Similarly, the y intercepts are found by setting x equal to 0 in the equation of the curve and solving for y

Figure 2.16 Curve Tracing



The graph of $f(x) = (x^2 + x + 1)/(x + 1)$, which has an asymptote at y=x. This may be approximated by curve sketching.

Determine the **symmetry** of the curve. If the exponent of x is always even in the equation of the curve then the y-axis is an axis of symmetry for the curve. Similarly, if the exponent of y is always even in the equation of the curve then the x-axis is an axis of symmetry for the curve. If the sum of the degrees of x and y in each term is always even or always odd, then the curve is symmetric about the origin and the origin is called a center of the curve.

Determine any bounds on the values of x and y. If the curve passes through the origin then determine the tangent lines there. For algebraic curves, this can be done by removing all but the terms of lowest order from the equation and solving. Similarly, removing all but the terms of highest order from the equation and solving gives the points where the curve meets the line at infinity.

Determine the asymptotes of the curve. Also determine from which side the curve approaches the asymptotes and where the asymptotes intersect the curve.

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Graphing on Computers and Calculators

Graphics can be created by hand, using computer programs, and with graphing calculators.

KEY POINTS

- A graphing calculator is a handheld scientific calculator capable of plotting graphs, solving simultaneous equations, and performing numerous other tasks with variables.
- Graphs can be created using open source and proprietary computer programs.
- High end features of computer programs include speed, high-resolution, and tree-dimensional graphing.

Graphics can be created by hand using simple everyday tools such as **graph** paper, pencils, markers, and rulers. However, today they are more often created using computer software, which is often both faster and easier. They can be created with graphing calculators.

Graphs are often created using computer software. Both open source computer and **proprietary** programs can be used for this purpose.

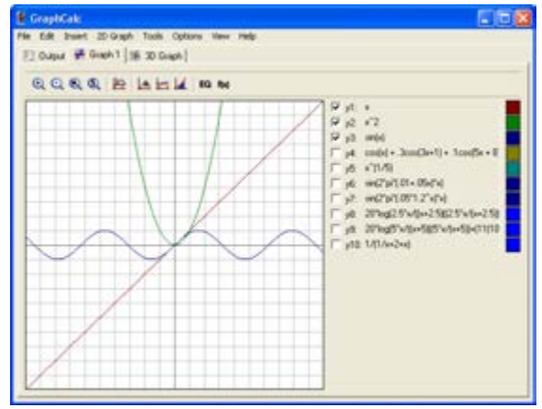

Figure 2.17
GraphCalc
Screenshot of GraphCalc

For example, GraphCalc (*Figure 2.17*) is an open source computer program that runs in Microsoft Windows and Linux that provides the functionality of a graphing calculator. GraphCalc includes many of the standard features of graphing calculators, but also includes some higher-end features.

1. High resolution: Graphing calculator screens have a resolution typically less than 120×90 pixels, whereas computer monitors typically display 1280x1024 pixels or more.
2. Speed: Modern computers are considerably faster than handheld graphing calculators.
3. Three-dimensional graphing: While high-end graphing calculators can graph in 3-D, GraphCalc benefits from modern computers' memory, speed, and graphics acceleration.

Mathematica is an example of proprietary computational software program used in scientific, engineering, and mathematical fields and other areas of technical computing. It also includes tools for visualizing and analyzing graphs.

A graphing calculator (*Figure 2.18*) typically refers to a class of handheld **scientific calculators** that are capable of plotting graphs, solving simultaneous equations, and performing numerous other tasks with variables. Most popular graphing calculators are also programmable, allowing the user to create customized programs, typically for scientific/engineering and education applications. Due to their large displays intended for graphing, they can also accommodate several lines of text and calculations at a time.

Figure 2.18 Graphing Calculator Calculators graph curves by drawing each pixel as a linear approximation of the function.

Some of the more recent graphing calculators are capable of color output, and also feature animated and interactive drawing of math plots (2D and 3D), other figures such as animated geometry theorems, preparation of documents which can include these plots and drawings, etc... This is giving the new graphing calculators a presence even in high school courses where they were formerly disallowed. Some calculator manufacturers also offer computer software for emulating and working with handheld graphing calculators.

Many graphing calculators can be attached to devices like electronic thermometers, pH gauges, weather instruments, decibel and light meters, accelerometers, and other sensors and therefore function as data loggers, as well as WiFi or other communication modules for monitoring, polling and interaction with the teacher. Student laboratory exercises with data from such devices enhances learning of math, especially statistics and mechanics.

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Optimization

Mathematical optimization is the selection of a best element (with regard to some criteria) from some set of available alternatives.

KEY POINTS

- Many design problems can also be expressed as optimization programs.
- Optimization relies heavily on finding maxima and minima. For this, calculus is useful.
- An example would be companies seeking to maximize sales while minimizing costs.

Mathematical **optimization** is the selection of a best element (with regard to some criteria) from some set of available alternatives. Optimization process that involves only a single variable is rather straightforward. After finding out the function f(x) to be optimized, local maxima or minima at critical points can be easily found. (Of course, end points may have maximum/minimum values as well.) The same strategy applies for optimization with several variables.

Many design problems can also be expressed as optimization programs. This application is called design optimization. One subset is the engineering optimization, and another recent and growing subset of this field is multidisciplinary design optimization, which, while useful in many problems, has in particular been applied to aerospace engineering problems.

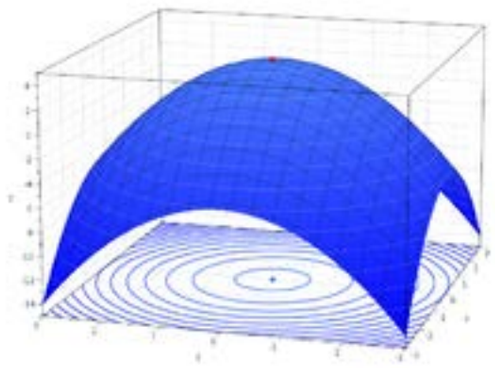

Figure 2.19 Maxima Finding maxima is useful in optimization problems.

Economics is closely linked to optimization of agents. Modern optimization theory includes traditional optimization theory but also overlaps with game theory and the study of economic equilibria. In microeconomics, the utility maximization problem and its dual problem, the expenditure minimization problem, are economic optimization problems. Insofar as they behave consistently, consumers are assumed to maximize their utility, while firms are usually assumed to maximize their profit. Also, agents are often modeled as being risk-averse, thereby preferring to avoid risk. Asset prices are also modeled using optimization theory, though the underlying mathematics relies on optimizing **stochastic** processes rather than on static optimization. Trade theory also uses optimization to explain trade patterns between nations. The optimization of market portfolios is an example of multi-objective optimization in economics.

Another field that uses optimization techniques extensively is operations research. Operations research also uses stochastic modeling and simulation to support improved decision-making. Increasingly, operations research uses stochastic programming to model dynamic decisions that adapt to events; such problems can be solved with large-scale optimization and stochastic optimization methods.

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Newton's Method

Newton's Method is a method for finding successively better approximations to the roots (or zeroes) of a real-valued function.

KEY POINTS

- Newton's method proceeds by an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line (which can be computed using the tools of calculus).
- Then compute the x-intercept of this tangent line (which is easily done with elementary algebra). This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.
- The more times you iterate, the more accurate the approximation to the actual roots.

In numerical analysis, Newton's method (also known as the Newton–Raphson method), named after Isaac Newton and Joseph Raphson, is a method for finding successively better approximations to the **roots** (or zeroes) of a real-valued function. In other words find x such that f(x) = 0. Also known as the x intercept.

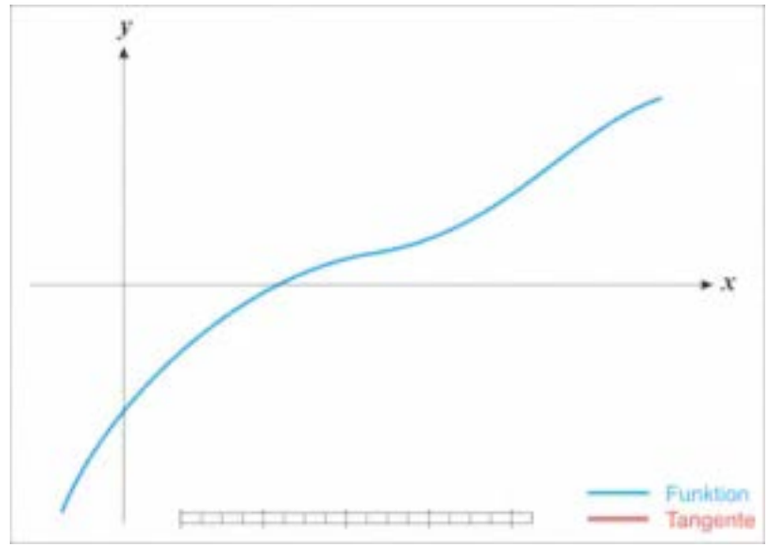
The Newton-Raphson method in one variable is implemented as follows:

Given a function f defined over the reals x, and its **derivative** f ', we begin with a first guess x0 for a root of the function f. Provided the function satisfies all the assumptions made in the derivation of the formula, a better approximation x1 is x0 - f(x0)/f'(x0). Geometrically, (x1, 0) is the intersection with the x-axis of a line **tangent** to f at (x0, f (x0)).The process is repeated as $x_n + 1 = x_n - f(x_n)/f'(x_n)$ until a sufficiently accurate value is reached.

This algorithm is first in the class of Householder's methods, succeeded by Halley's method. The method can also be extended to complex functions and to systems of equations.

The idea of the method is as follows: one starts with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line (which can be computed using the tools of calculus), and one computes the x-intercept of this tangent line (which is easily done with elementary algebra). This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated (*Figure 2.20*).

Figure 2.20 Newton's Method



The function f is shown in blue and the tangent line is in red. We see that xn+1 is a better approximation than xn for the root x of the function f.

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Concavity and the Second Derivative Test

The second derivative test is a criterion for determining whether a given critical point is a local maximum or a local minimum.

KEY POINTS

- A critical point is a point where the derivative is 0.
- If the second derivative is positive, the point is a minimum.
- If the second derivative is negative, the point is a maximum.
- If the second derivative is 0, the test is inconclusive.

In calculus, the second derivative test is a criterion for determining whether a given **critical point** of a real function of one variable is a **local maximum** or a **local minimum** using the value of the second derivative at the point (*Figure 2.21*).

The test states: if the function f is twice differentiable at a critical point x (i.e. f'(x) = 0), then:

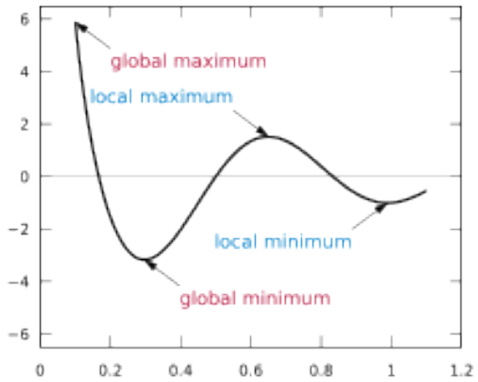
If f''(x) < 0 then f(x) has a local maximum at x.

If f''(x) > 0 then f(x) has a local minimum at x.

If f(x) = 0, the test is inconclusive.

In the latter case, Taylor's Theorem may be used to determine the behavior of f near x using higher derivatives.

Figure 2.21 Maxima and Minima



Telling whether a critical point is a maximum or a minimum has to do with the second derivative. If it is concave up at the point, it is a minimum; if concave down, maximum.

Proof of the Second Derivative Test:

Suppose we have f''(x) > 0 (the proof for f''(x) < 0 is analogous). By assumption, f'(x)=0. Then,

$$0 < f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Thus, for h sufficiently small h we get

$$\frac{f'(x+h)}{h} > 0$$

which means that f'(x+h) < 0 if h < 0 (intuitively, f is decreasing as it approaches x from the left), and that f'(x+h) > 0 if h > 0 (intuitively, f is increasing as we go right from x). Now, by the first derivative test, f(x) has a local minimum at x.

A related but distinct use of second derivatives is to determine whether a function is concave up or concave down at a point. It does not, however, provide information about inflection points. Specifically, a twice-differentiable function f is concave up if f''(x) is positive and concave down if f''(x) is negative.

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Differentials

Differentials are the principal part of the change in a function y = f(x) with respect to changes in the independent variable.

KEY POINTS

- Differentials are notated by dx or dy.
- They represent an **infinitesimal** increase in the variable x or y.
- Higher order differentials represent successive derivatives.

In calculus, the differential represents the principal part of the change in a function y = f(x) with respect to changes in the independent variable (*Figure 2.22*). The differential dy is defined by $d^2(y) = f''(x)(dx)^2 + f'(x)(d^2x)$ and so forth.

$$dy = f'(x)dx,$$

where f'(x) is the derivative of f with respect to x, and dx is an additional real variable (so that dy is a function of x and dx). The notation is such that the equation

$$dy = \frac{dy}{dx}dx$$

holds, where the derivative is represented in the Leibniz notation dy/dx, and this is consistent regarding the derivative as the quotient of the differentials. One also writes,

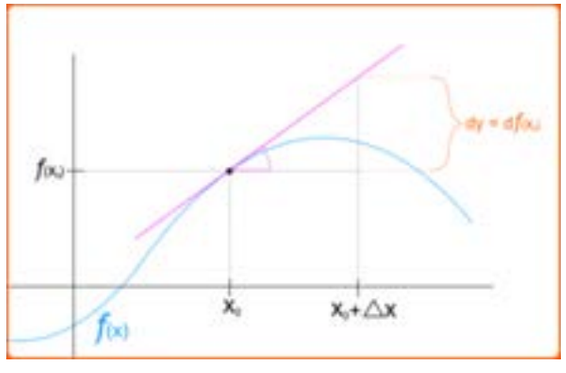

Figure 2.22 Differentials The differential of a function $f(x)$ at a point x0.

$$df(x) = f'(x)dx.$$

The precise meaning of the variables dy and dx depends on the context of the application and the required level of mathematical rigor. The domain of these variables may take on a particular geometrical significance if the differential is regarded as a particular differential form, or a particular analytical significance if the differential is regarded as a linear approximation to the increment of a function. In physical applications, the variables dx and dy are often constrained to be very small ("infinitesimal").

Higher-order differentials of a function y = f(x) of a single variable x can be defined via

$$d(dy) = d(f'(x))$$

and, in general,

$$d^n(y) = f^{(n)}(x)(dx)^n$$

Informally, this justifies Leibniz's notation for higher-order derivatives.

When the independent variable x itself is permitted to depend on other variables, then the expression becomes more complicated, as it must also include higher order differentials in x itself. Thus, for instance,

$$d^2(y) = f''(x)(dx)^2 + f'(x)(d^2x)$$

and so forth.

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Section 3

Integrals

Antiderivatives

Area and Distances

The Definite Integral

The Fundamental Theorem of Calculus

Indefinite Integrals and the Net Change Theorem

The Substitution Rule

Further Transcendental Functions

Numerical Integration

https://www.boundless.com/calculus/derivatives-and-integrals/integrals/

Antiderivatives

An antiderivative is a differentiable function F whose derivative is equal to f (i.e., F′ = f).]

KEY POINTS

- The process of solving for antiderivatives is called antidifferentiation, and its opposite operation is called differentiation, which is the process of finding a derivative.
- Antiderivatives are related to definite integrals through the fundamental theorem of calculus: the definite integral of a function over an interval is equal to the difference between the values of an antiderivative evaluated at the endpoints of the interval.
- The graphs of antiderivatives of a given function are vertical translations of each other, with each graph's location depending upon the value of constant C.

An antiderivative is a differentiable function F whose **derivative** is equal to f (i.e., F′ = f). The process of solving for antiderivatives is called antidifferentiation, and its opposite operation is called differentiation, which is the process of finding a derivative. Antiderivatives are related to **definite integrals** through the fundamental theorem of calculus: the definite integral of a function over an interval is equal to the difference between the values of an antiderivative evaluated at the endpoints of the interval.

Let's consider the case of function $F(x) = x^3/3$, which is an antiderivative of $f(x) = x^2$. As the derivative of a constant is zero, x^2 will have an infinite number of antiderivatives, such as $(x^3/3) + 0$, $(x^3/3) + 7$, $(x^3/3) - 42$, $(x^3/3) + 293$, etc... Thus, all the antiderivatives of x2 can be obtained by adding the value of C in $F(x) = (x^3/3) + C$, where C is an arbitrary constant known as the constant of integration. Essentially, the graphs of antiderivatives of a given function are vertical translations of each other, with each graph's location depending upon the value of C (*Figure 2.23*).

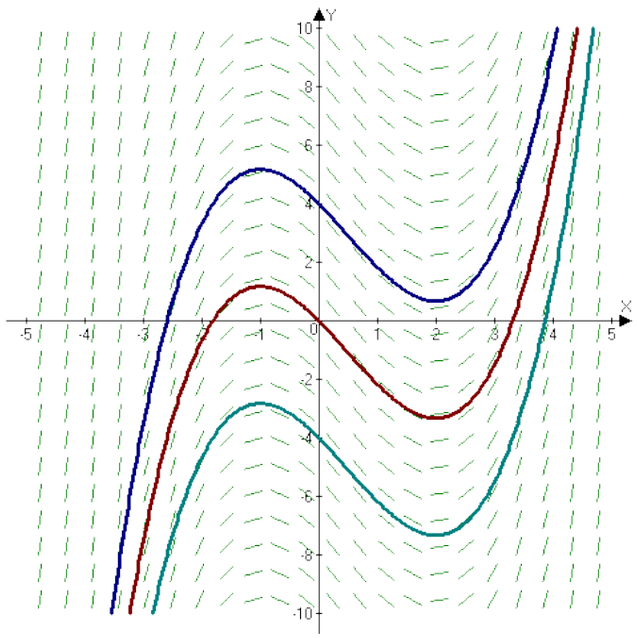

Figure 2.23
Slope Field
The slope field of $F(x) = (x^3/3)-(x^2/2)-x+c$, showing three of the infinitely many solutions that can be produced by varying the arbitrary constant C.

Antiderivatives are important because they can be used to compute definite integrals with the fundamental theorem of calculus: if F is an antiderivative of the integrable function f, and f is continuous over the interval [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Because of this rule, each of the infinitely many antiderivatives of a given function f is sometimes called the "general integral" or "indefinite integral" of f, and is written using the integral symbol with no bounds:

$$\int f(x)dx.$$

If F is an antiderivative of f, and the function f is defined on some interval, then every other antiderivative G of f differs from F by a constant: there exists a number C such that G(x) = F(x) + C for all x. C is called the arbitrary constant of integration. If the domain of F is a disjoint union of two or more intervals, then a different constant of integration may be chosen for each of the intervals. For instance,

$$F(x) = \begin{cases} -1/x + C_1 if x < 0 \\ -1/x + C_2 if x > 0 \end{cases}$$

is the most general antiderivative of $f(x) = 1/x^2$ on its natural domain, $(-\infty; 0) \bigcup (0; \infty)$.

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Area and Distances

Defined integrals are used in many practical situations that require distance, area, and volume calculations.

KEY POINTS

- The definite integral $\int_a^b f(x)dx$ is defined informally to be the area of the region in the xy-plane bound by the graph of f, the x-axis, and the vertical lines x = a and x = b, such that the area above the x-axis adds to the total, and the area below the x-axis subtracts from the total.
- According to the fundamental theorem of calculus: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an antiderivative F of f is known, the definite integral of f over that interval is given by $\int_a^b f(x)dx = F(b) - F(a)$.
- When practical approximation does not provide precise enough results for distance, area, and volume calculations, integration must be performed.

Integration is an important concept in mathematics and--together with its inverse, differentiation-- is one of the two main operations in calculus. Given a function f of a real variable x and an interval [a, b] of the real line, the **definite integral**.

Integration is connected with differentiation through the fundamental theorem of calculus: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an **antiderivative** F of f is known, the definite integral of f over that interval is given by: $\int_a^b f(x)dx = F(b) - F(a)$. Definite integrals appear in many practical situations that require distance, area, and volume calculations.

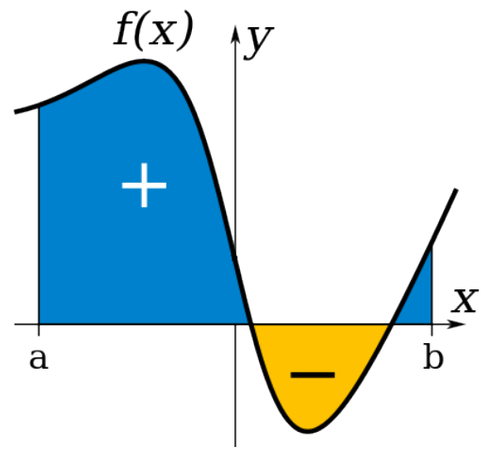


Figure 2.24
Definite Integral
A definite integral of a function can be represented as the signed area of the region bound by its graph.

Area

To start off, consider the curve y = f(x) between x = 0 and x = 1 with $f(x) = \sqrt{x}$. We ask: What is the area under the function f, in the interval from 0 to 1? and call this (yet unknown) area the integral of f. The notation for this integral will be $\int_0^1 \sqrt{x}dx$.

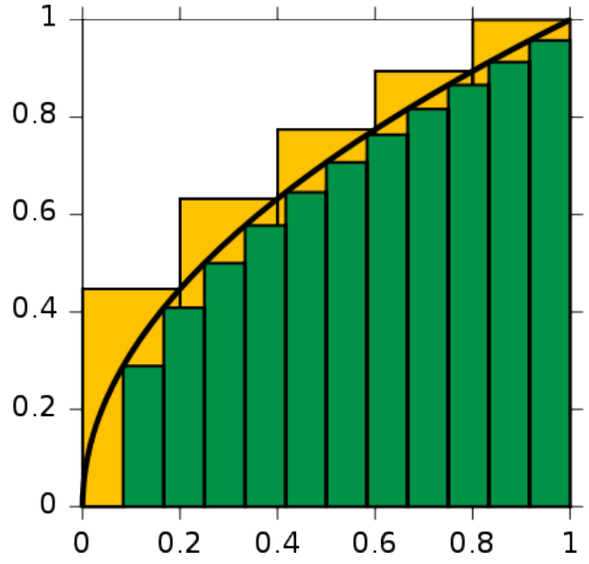
As a first approximation, look at the unit square given by the sides x = 0 to x = 1, y = f(0) = 0, and y = f(1) = 1. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles will yield a better result, so we cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on, up to 1. Fit a box for each step using the right end height of each curve piece; thus, $\sqrt{\frac{1}{5}}, \sqrt{\frac{2}{5}}$, and so on, up to $\sqrt{1} = 1$. Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

$$\sqrt{\frac{1}{5}}(\frac{1}{5} - 0) + \sqrt{\frac{2}{5}}(\frac{2}{5} - \frac{1}{5}) + \ldots + \sqrt{\frac{5}{5}}(\frac{5}{5} - \frac{4}{5}) \approx 0.7497.$$

Notice that we are taking a sum of finitely many function values of f, multiplied with the differences of two subsequent approximation points. We can easily see that the approximation is still too large. Using more steps produces a closer approximation, but will never be exact: replacing the 5 subintervals by twelve as depicted, we will get an approximate value for the area of 0.6203, which is too small (see *Figure 2.25*). The key idea is the transition from adding a finite number of differences of approximation points multiplied by their respective function values to using an infinite number of fine, or infinitesimal, steps.

Using the fundamental theorem of calculus to the square root curve, $f(x) = x^{1/2}$, we look at the antiderivative $F(x) = (2/3)x^{3/2}$, and simply take F(1) – F(0), where 0 and 1 are the boundaries of the interval [0,1]. So the exact value of the area under the curve is computed formally as

Figure 2.25 Integral Approximation



Approximations to integral of √x from 0 to 1, with yellow 5 right samples (above) and green 12 left samples (below).

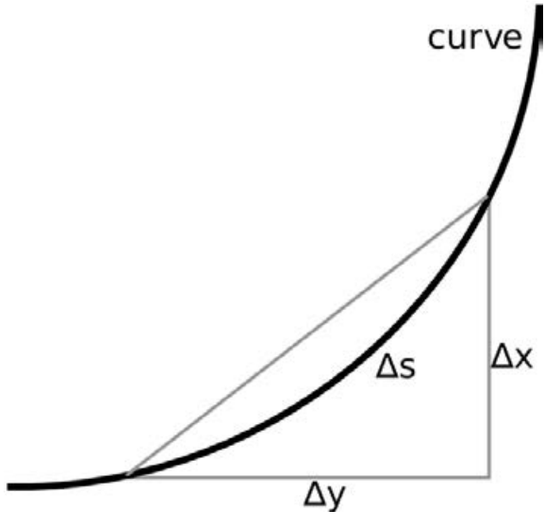
$$\int_0^1 \sqrt{x}dx = \int_0^1 x^{1/2}dx = F(1) - F(0) = \frac{2}{3}.$$

Distance (Finding arc length by Integrating)

If you know the velocity v(t) of an object as a function of time, you can simply integrate v(t) over time to calculate the distance the object traveled. Since this is equivalent to evaluating the area under the curve v(t), we will not discuss more on this.

However, you can also use integrals to calculate length--for example, the length of an arc described by a function y = f(x).

Figure 2.26 Calculating arc length



For a small piece of curve, Δs can be approximated with the Pythagorean theorem.

Consider an infinitesimal part of the curve ds on the curve (or consider this as a limit in which the change in s approaches ds). According to Pythagoras's theorem, $ds^2 = dx^2 + dy^2$ (*Figure 2.26*), from which

$$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. Equivalently, if a curve is defined parametrically by x = X(t) and y = Y(t), we get

$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt$; then, its arc length between t = a and t = b is $s = \int_a^b \sqrt{[X'(t)]^2 + [Y'(t)]^2}\, dt$.

Example: For a curve described by a parameter t: $\begin{cases} y = t^5 \\ x = t^3 \end{cases}$. the arc length integral for values of t from -1 to 1 is

$$int_{-1}^{1} \sqrt{(3t^2)^2 + (5t^4)^2}\, dt = \int_{-1}^{1} \sqrt{9t^4 + 25t^8}\, dt.$$

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The Definite Integral

A definite integral is the area of the region in xy-plane bound by graph of f, x-axis, and vertical lines x = a and x = b.

KEY POINTS

- Integration is an important concept in mathematics and--together with its inverse, differentiation-- is one of the two main operations in calculus.
- Integration is connected with differentiation through the fundamental theorem of calculus: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an antiderivative F of f is known, the definite integral of f over that interval is given by $\int_a^b f(x)dx = F(b) - F(a)$.
- Definite integrals appear in many practical situations, and their actual calculation is important in the type of precision engineering (of any discipline) that requires exact and rigorous values.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration is an important concept in mathematics and--together with its inverse, differentiation-- is one of the two main operations in calculus. Given a function f of a real variable x and an interval [a, b] of the real line, the **definite integral** $\int_a^b f(x)dx$ is defined informally to be the area of the region in the xy-plane bound by the graph of f, the x-axis, and the vertical lines x = a and x = b, such that the area above the x-axis adds to the total, and that the area below the x-axis subtracts from the total (*Figure 2.27*).

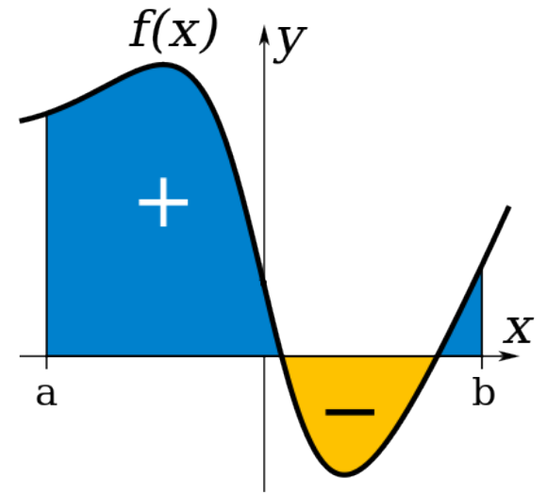

Figure 2.27 Definite Integral
A definite integral of a function can be represented as the signed area of the region bounded by its graph.

The integrals discussed in this atom are termed definite integrals.

The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an **antiderivative** F of f is known, the definite integral of f over that interval is given by

$$\int_a^b f(x)dx = F(b) - F(a).$$

Definite integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but precision engineering (of any discipline) requires exact and rigorous values for these elements.

For example, consider the curve y = f(x) between x = 0 and x = 1 with $f(x) = \sqrt{x}$.

We ask, "What is the area under the function f, in the interval from 0 to 1?" and call this (yet unknown) area the integral of f. The notation for this integral will be $\int_0^1 \sqrt{x}dx$.

As a first approximation, look at the unit square given by the sides x = 0 to x = 1, y = f(0) = 0, and y = f(1) = 1. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles should yield a better result, so we will cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on, up to 1. Fit a box for each step using the right end height of each curve piece, thus obtaining $\sqrt{\frac{1}{5}}, \sqrt{\frac{2}{5}}$, and so on, up to $\sqrt{1} = 1$. Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

$$\sqrt{\frac{1}{5}}(\frac{1}{5} - 0) + \sqrt{\frac{2}{5}}(\frac{2}{5} - \frac{1}{5}) + \ldots + \sqrt{\frac{5}{5}}(\frac{5}{5} - \frac{4}{5}) \approx 0.7497.$$

Notice that we are taking a finite sum of many function values of f, multiplied with the differences of two subsequent approximation points. We can easily see that the approximation is still too large. Using more steps produces a closer approximation, but will never be exact: replacing the 5 subintervals by twelve as depicted, we will get an approximate value for the area of 0.6203, which is too small. The key idea is the transition from adding a finite number of differences of approximation points multiplied by their respective function values to using an infinite number of fine, or infinitesimal, steps.

As for the actual calculation of integrals, the fundamental theorem of calculus, due to Newton and Leibniz, is the fundamental link between the operations of differentiating and integrating. Applied to the square root curve, $f(x) = x^{1/2}$, the theorem says to look at the antiderivative, $F(x) = (2/3)x^{3/2}$, and simply take F(1) – F(0), where

0 and 1 are the boundaries of the interval [0,1]. So the exact value of the area under the curve is computed formally as

$$\int_0^1 \sqrt{x}dx = \int_0^1 x^{1/2}dx = F(1) - F(0) = \frac{2}{3}.$$

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The Fundamental Theorem of Calculus

The fundamental theorem of calculus is a theorem that links the concept of the derivative of a function to the concept of the integral.

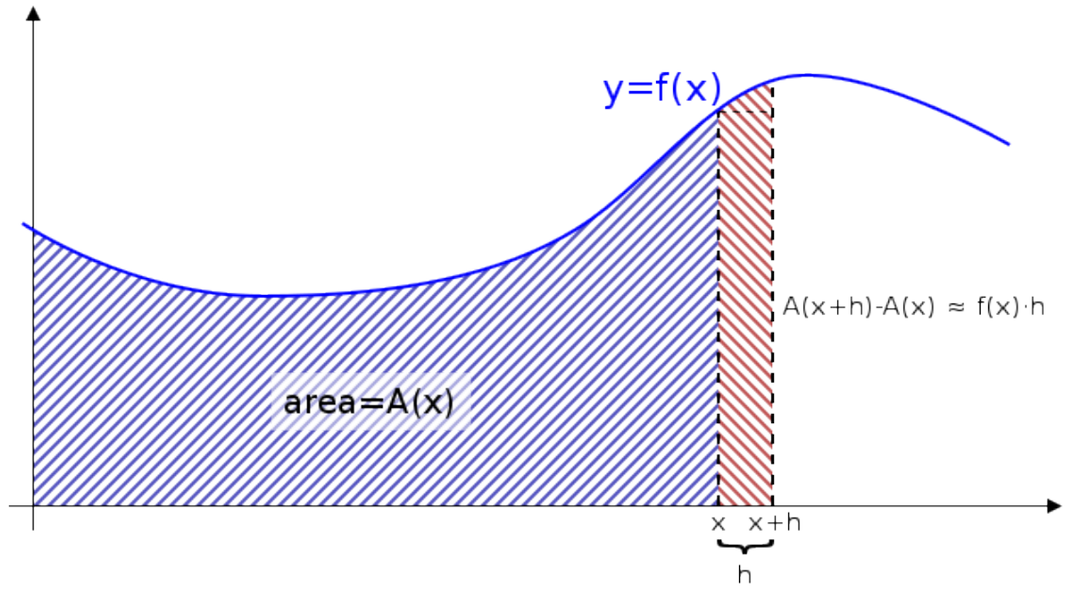
KEY POINTS

- The first part of the theorem shows that an indefinite integration can be reversed by differentiation.
- The second part allows one to compute the definite integral of a function by using any one of its infinitely many antiderivatives.
- The second part of the theorem has invaluable practical applications because it markedly simplifies the computation of definite integrals.

The fundamental theorem of calculus is a theorem that links the concept of the **derivative** of a function to the concept of the integral (*Figure 2.28*).

There are two parts to the theorem. Loosely put, the first part deals with the derivative of an antiderivative, while the second part deals with the relationship between **antiderivatives** and **definite integrals**.

Figure 2.28 The Fundamental Theorem of Calculus



We can see from this picture that the Fundamental Theorem of Calculus works. By definition, the derivative of A(x) is equal to [A(x+h)–A(x)]/h as h tends to zero. By replacing the numerator, A(x+h)–A(x), by hf(x) and dividing by h, f(x) is obtained. Taking the limit as h tends to zero completes the proof of the Fundamental Theorem of Calculus.

The first part of the theorem, sometimes called the first fundamental theorem of calculus, shows that an indefinite integration can be reversed by differentiation. This part of the theorem is also important because it guarantees the existence of antiderivatives for continuous functions.

Let f be a continuous real-valued function defined on a closed interval [a, b]. Let F be the function defined, for all x in [a, b], by

$$F(x) = \int_a^b f(t)dt.$$

Now, F is continuous on [a, b], differentiable on the open interval (a, b), and F'(x) = f(x) for all x in (a, b).

The second part, sometimes called the second fundamental theorem of calculus, allows one to compute the definite integral of a function by using any one of its infinitely many antiderivatives. This part of the theorem has invaluable practical applications, because it markedly simplifies the computation of definite integrals.

Let f and F be real-valued functions defined on a closed interval [a, b] such that the derivative of F is f. That is, f and F are functions such that, for all x in [a, b], F'(x) = f(x). If f is a Riemann integrable on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$.

The first published statement and proof of a restricted version of the fundamental theorem was by James Gregory (1638–1675). Isaac Barrow (1630–1677) proved a more generalized version of the theorem, while Barrow's student Isaac Newton (1643–1727) completed the development of the surrounding mathematical theory. Gottfried Leibniz (1646–1716) systematized the knowledge into a calculus for infinitesimal quantities and introduced the notation used today.

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Indefinite Integrals and the Net Change Theorem

Indefinite integral is defined as $\int f(x)dx = F(x) + C$, where F satisfies F'(x) = f(x) and C is any constant.

KEY POINTS

- Constant Rule for indefinite integrals: $\int cf(x)dx = c\int f(x)dx$.
- Sum/Difference Rule for indefinite integrals: $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$.
- Integral of a rate of change is the net change (displacement for position functions): $\int_a^b f(x)dx = f(b) - f(a)$.

As you remember from the atoms on **antiderivatives**, F is said to be an antiderivative of f if F'(x) = f(x). However, F is not the only antiderivative. We can add any constant (C) to F without changing the derivative. With this in mind, we define the indefinite integral as follows: $\int f(x)dx = F(x) + C$, where F satisfies F'(x) = f(x) and C is any constant.

f(x), the function being integrated, is known as the integrand. Note that the indefinite **integral** yields a family of functions.

For example, function $F(x) = x^3/3$ is an antiderivative of $f(x) = x^2$. As the derivative of a constant is zero, x^2 will have an infinite number of antiderivatives, such as $(x^3/3) + 0$, $(x^3/3) + 7$, $(x^3/3) - 42$, $(x^3/3) + 293$, etc. Thus, all the antiderivatives of x2 can be obtained by changing the value of C in $F(x) = (x^3/3) + C$, where C is an arbitrary constant known as the constant of integration. Essentially, the graphs of antiderivatives of a given function are vertical translations of each other, with each graph's location depending upon the value of C (see *Figure 2.29*).

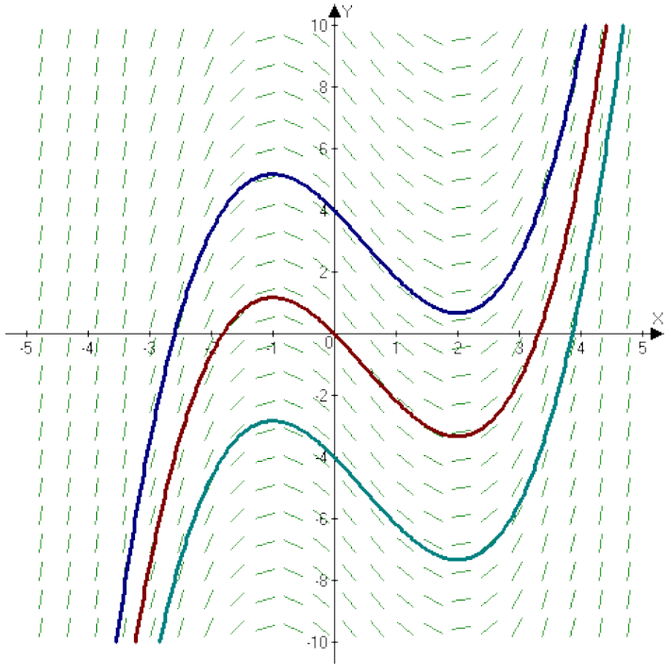

Figure 2.29 Slope Field

The slope field of F(x) = (x^3/3)-(x^2/2)-x+c, showing three of the infinitely many solutions that can be produced by varying the arbitrary constant C.

Indefinite integrals exhibit the following basic properties.

a) Constant Rule for indefinite integrals: $\int cf(x)dx = c\int f(x)dx.$

b) Sum/Difference Rule for indefinite integrals:

$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$ and

$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx.$

When integrating over a specified domain, we speak of a definite integral. Integrating over a domain D is written as $\int_a^b f(x)dx$ if the domain is an interval [a, b] of x. Such a problem can be solved using the net change theorem, which states that the integral of a rate of change is the net change (displacement for position functions): $\int_a^b f(x)dx = f(b) - f(a)$. Basically, the theorem states that the integral of f or F' from a to b is the area between a and b, or the difference in area from the position of f(a) to f(b). This can be applied to things such as volume, concentration, density, population, cost, and velocity.

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The Substitution Rule

Integration by substitution is an important tool for mathematicians used to find integrals and antiderivatives.

KEY POINTS

- The substitution x = g(t) yields dx/dt = g'(t) and thus, formally, dx = g'(t)dt, which is the required substitution for dx.
- U-substitution (w-substitution) is used to simplify a given integral.
- Substitution can be used to determine antiderivatives.

Integration by substitution, also known as u-substitution, is a method for finding integrals (*Figure 2.30*). Using the fundamental theorem of calculus often requires finding an **antiderivative**.

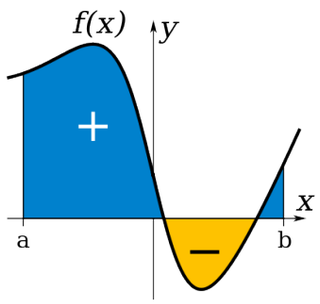

Figure 2.30
Definite Integral
A definite integral of a function can be represented as the signed area of the region bounded by its graph.

For this and other reasons, integration by substitution is an important tool for mathematicians. It is the counterpart to the chain rule of differentiation.

Let $I \subseteq \mathrm{R}$ be an interval and $g : [a, b] \to 1$ be a continuously differentiable function. Suppose that $f : I \to \mathrm{R}$ is a continuous function. Then $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt$.

Using Leibniz notation: The substitution x = g(t) yields dx/dt = g'(t) and thus, formally, dx = g'(t)dt, which is the required substitution for dx.

The formula is used to transform one integral into another integral that is easier to compute. Thus, the formula can be used from left to right or from right to left in order to simplify a given integral. When used in the latter manner, it is sometimes known as u-substitution or w-substitution.

For example, consider the integral $\int_0^2 x\cos(x^2+1)$. If we make the substitution $u = x^2 + 1$, we obtain $du = 2x\,dx$ and

$$\int_{x=0}^{x=2} x\cos(x^2+1) = \frac{1}{2}\int_{u=1}^{u=5} \cos(u)du = \frac{1}{2}(\sin(5) - \sin(1)).$$

It is important to note that since the lower limit x = 0 was replaced with u = 02 + 1 = 1, and the upper limit x = 2 replaced with u = 22 + 1 = 5, a transformation back into terms of x was unnecessary.

Substitution can be used to determine antiderivatives if one chooses a relation between x and u, determines the corresponding relation between dx and du by differentiating, and performs the substitutions. An antiderivative for the substituted function can hopefully be determined; the original substitution between u and x is then undone.

Similar to our first example above, we can determine the following antiderivative with this method:

$$\int x\cos(x^2+1) = \frac{1}{2}\int 2x\cos(x^2+1)dx = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(u) + C = \frac{1}{2}sin(x^2+1) + C$$

where C is an arbitrary constant of integration. Note that there were no integral boundaries to transform, but in the last step we had to revert the original substitution, $u = x^2 + 1$.

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Further Transcendental Functions

A transcendental function is a function that is not algebraic.

KEY POINTS

- Transcendental functions cannot be expressed as a solution of a polynomial equation whose coefficients are themselves polynomials with rational coefficients.
- Examples of transcendental functions include the exponential function, the logarithm, and the trigonometric functions.
- Transcendental functions can be an easy-to-spot source of dimensional errors.

A transcendental function is a function that is not algebraic. Such a function cannot be expressed as a solution of a **polynomial** equation whose coefficients are themselves polynomials with rational coefficients. Examples of transcendental functions include the **exponential function** (*Figure 2.31*), the logarithm (*Figure 2.32*), and the **trigonometric functions** (*Figure 2.33*).

Formally, an analytic function $f(z)$ of the real or complex variables $z_1,\ldots,z_n$ is transcendental if $z_1, \ldots, z_n, f(z)$ are algebraically

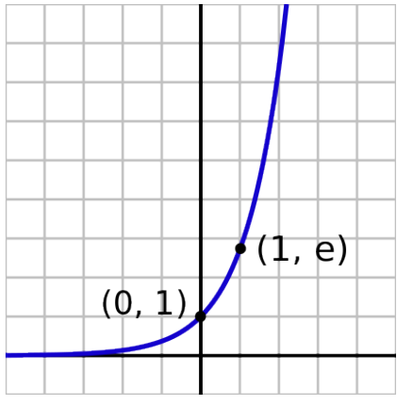


Figure 2.31 Natural Exponential Function

The natural exponential function y = e^x.

independent, i.e., if f is transcendental over the field $C(z_1, \ldots, z_n)$.

A transcendental function is a function that "transcends" algebra in the sense that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, power, and root extraction.

The following functions are transcendental:

$f_1(x) = x^{\Pi}$

$f_2(x) = c^x, x \neq 0,1$

$f_3(x) = x^x$

$f_4(x) = x^{\frac{1}{x}}$

$f_5(x) = log_c(x)$

$f_6(x) = sin(x)$

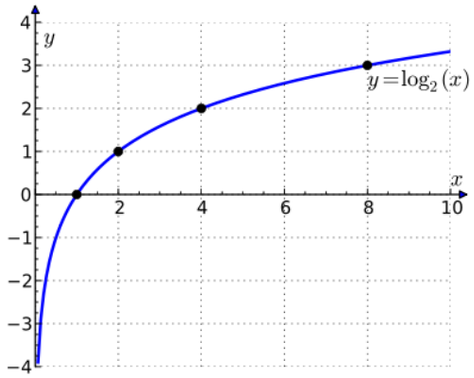
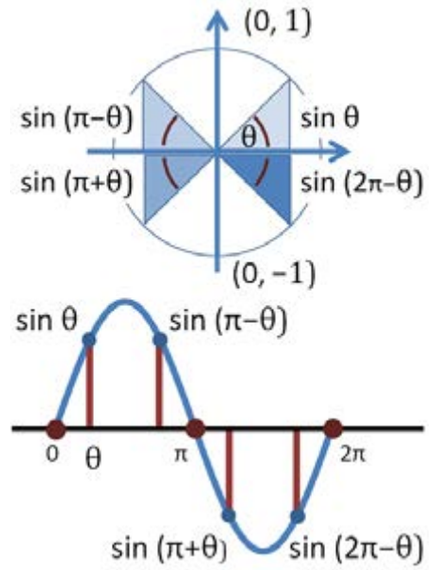


Figure 2.32 Log Base 2 Function

The graph of the logarithm to base 2.

Note that, for f_2 in particular, if we set c equal to e, the base of the natural logarithm, then we find that e^x is a transcendental function. Similarly, if we set c equal to e in f_5, then we find that ln(x), the natural logarithm, is a transcendental function.

Figure 2.33 Trigonometric Functions



(Top): Trigonometric function sinθ for selected angles θ, π – θ, π + θ, and 2π – θ in the four quadrants. (Bottom) Graph of sine function versus angle. Angles from the top panel are identified.

In dimensional analysis, transcendental functions are notable because they make sense only when their argument is dimensionless (possibly after algebraic reduction). Because of this, transcendental functions can be an easy-to-spot source of dimensional errors. For example, log(5 meters) is a nonsensical expression, unlike log(5 meters/3 meters) or log(3) meters. One could attempt to apply a logarithmic identity to get log(10) + log(m), which highlights the problem: applying a non-algebraic operation to a dimension creates meaningless results.

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Numerical Integration

Integration--together with its inverse, differentiation-- is one of the two main operations in calculus.

KEY POINTS

- Integration is connected with differentiation through the the fundamental theorem of calculus. If f is a continuous real-valued function defined on a closed interval [a, b], then, once an antiderivative F of f is known, the definite integral of f over that interval is given by $\int_a^b f(x)dx = F(b) - F(a)$.
- The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century.
- Defined integrals appear in many practical situations and are important for precision engineering, which requires exact and rigorous values for these elements.

Integration is an important concept in mathematics and--together with its inverse, **differentiation**-- is one of the two main operations in calculus. Given a function f of a real variable x and an interval [a, b] of the real line, the **definite integral**

$$\int_a^b f(x)dx$$

is defined informally to be the area of the region in the xy-plane bound by the graph of f, the x-axis, and the vertical lines x = a and x = b, such that the area above the x-axis adds to the total, and that the area below the x-axis subtracts from the total.

The integrals discussed in this atom are termed definite integrals (*Figure 2.34*).

The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an antiderivative F of f is known, the definite integral of f over that interval is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$

Defined integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but precision

engineering (of any discipline) requires exact and rigorous values for these elements.

For example, consider the curve y = f(x) between x = 0 and x = 1 with

$$f(x) = \sqrt{x}$$

We ask:

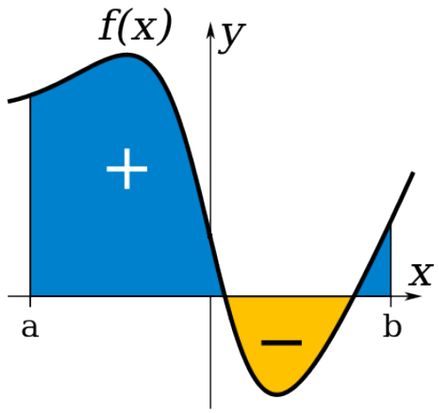

Figure 2.34
Definite Integral
A definite integral of a function can be represented as the signed area of the region bounded by its graph.

What is the area under the function f, in the interval from 0 to 1?

and call this (yet unknown) area the integral of f. The notation for this integral will be

$$\int_0^1 \sqrt{x}\,dx$$

As a first approximation, look at the unit square given by the sides x = 0 to x = 1, y = f(0) = 0, and y = f(1) = 1. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles will yield a better result, so we cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on, up to to 1. Fit a box for each step using the right end height of each curve piece; thus $\sqrt{\frac{1}{5}}, \sqrt{\frac{2}{5}}$, and so on, up to $\sqrt{1} = 1$. Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

$$\sqrt{\frac{1}{5}}(\frac{1}{5} - 1) + \sqrt{\frac{2}{5}}(\frac{2}{5} - \frac{1}{5}) + \ldots + \sqrt{\frac{5}{5}}(\frac{5}{5} - \frac{4}{5}) \approx 0.7497.$$

Notice that we are taking a sum of finitely many function values of f, multiplied with the differences of two subsequent approximation points. We can easily see that the approximation is still too large. Using more steps produces a closer approximation, but will never be exact: replacing the 5 subintervals by twelve as depicted, we will get an approximate value for the area of 0.6203, which is too small. The key idea is the transition from adding a finite number of differences of approximation points multiplied by their respective function values to using an infinite number of fine, or infinitesimal, steps.

As for the actual calculation of integrals, the fundamental theorem of calculus--credited to Newton and Leibniz-- is the fundamental link between the operations of differentiating and integrating. Applied to the square root curve, $f(x) = x^{1/2}$, it says to look at the antiderivative $F(x) = (2/3)x^{3/2}$ and simply take F(1) – F(0), where 0 and 1 are the boundaries of the interval [0,1]. So the exact value of the area under the curve is computed formally as

$$\int_0^1 \sqrt{x}dx = \int_0^1 x^{1/2}dx = F(1) - F(0) = \frac{2}{3}.$$

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Section 4

Applications of Integration

Area Between Curves

Volumes

Average Value of a Function

Cylindrical Shells

Work

Volumes of Revolution

https://www.boundless.com/calculus/derivatives-and-integrals/applications-of-integration/

Area Between Curves

The area between the graphs of two functions is equal to the integral of a function, f(x), minus the integral of the other function, g(x): $A = \int_a^b (f(x) - g(x))\,dx$.

KEY POINTS

- Area is a quantity that expresses the extent of a two-dimensional surface or shape, or planar lamina, in the plane.
- The area between the graphs of two functions is equal to the integral of one function, f(x), minus the integral of the other function, g(x): $A = \int_a^b (f(x) - g(x))\,dx$ where f(x) is the curve with the greater y-value.
- The area between a positive-valued curve and the horizontal axis, measured between two values, a and b, (b>a) on the horizontal axis, is given by the integral from a to b of the function that represents the curve: $A = \int_a^b f(x)\,dx$.

Area is a quantity that expresses the extent of a two-dimensional surface or shape, or planar lamina, in the plane. Area can be understood as the amount of material with a given thickness that would be necessary to fashion a model of the shape, or the amount of paint necessary to cover the surface with a single coat. It is the two-dimensional analog of the length of a **curve** (a one-dimensional concept) or the volume of a solid (a three-dimensional concept).

Area Between Curves

The area between a positive-valued curve and the horizontal **axis**, measured between two values a and b (b is defined as the larger of the two values) on the horizontal axis, is given by the integral from a to b of the function that represents the curve shown in *Figure 2.35*. The area between the graphs of two functions is equal to the integral of one function, f(x), minus the integral of the other function, g(x):

$A = \int_a^b (f(x) - g(x))\,dx$ where f(x) is the curve with the greater y-value (*Figure 2.36*).

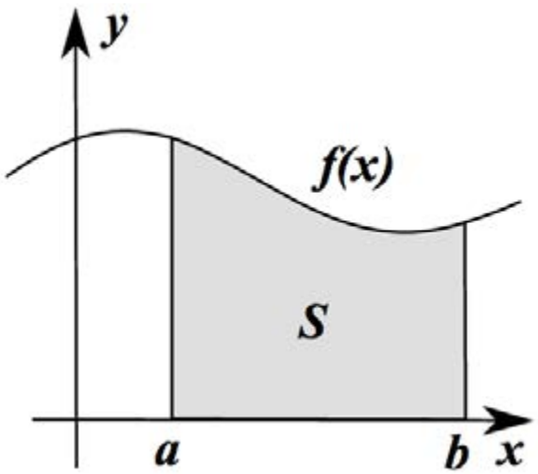

Figure 2.35
Integration: Area Under a Curve
Integration can be thought of as measuring the area under a curve, defined by f(x), between two points (here a and b).

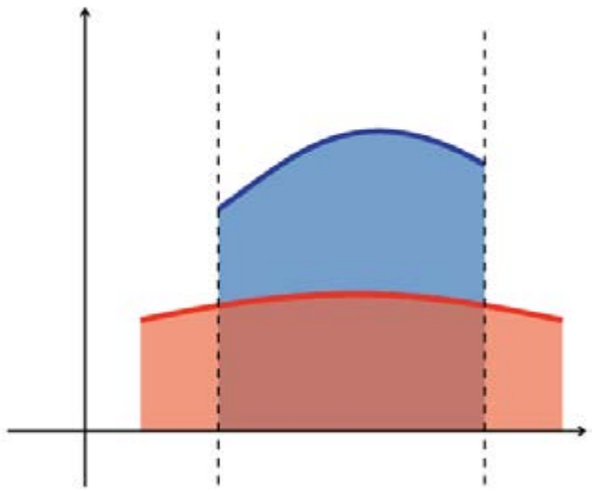

Figure 2.36 Area Between Two Graphs

The area between two graphs can be evaluated by calculating the difference between the integrals of the two functions.

Example

Find the area between the two curves f(x)=x and g(x)=0.5*x² between the interval x=0 and x=2.

Two curves, y=x and y=0.5*x², meet at the points $(x_0,y_0)=(0,0)$ and $(x_1,y_1)=(2,2)$. Since x>0.5*x² in the interval x=0 and x=2, the area can be calculated as the following:

$$A = \int_0^2 (x - \frac{1}{2}x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_{x=0}^{x=2} = \frac{2}{3}.$$

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Volumes

Volumes of complicated shapes can be calculated using integral calculus if a formula exists for the shape's boundary.

KEY POINTS

- Volume is the quantity of three-dimensional space enclosed by some closed boundary-- for example, the space that a substance or shape occupies or contains.
- Volumes of some simple shapes, such as regular, straight-edged, and circular shapes can be easily calculated using arithmetic formulas.
- Volumes of complicated shapes can be calculated using a triple integral of the constant function 1:
  $Vol(D) = \iiint_D dx\,dy\,dz.$

Volume is the quantity of three-dimensional space enclosed by some closed boundary-- for example, the space that a substance or shape occupies or contains. Three dimensional mathematical shapes are also assigned volumes. Volumes of some simple shapes, such as regular, straight-edged, and circular shapes can be easily calculated using arithmetic formulas. The volumes of more complicated shapes can be calculated using integral calculus if a

formula exists for the shape's boundary. One-dimensional figures (such as lines) and two-dimensional shapes (such as squares) are assigned zero volume in three-dimensional space.

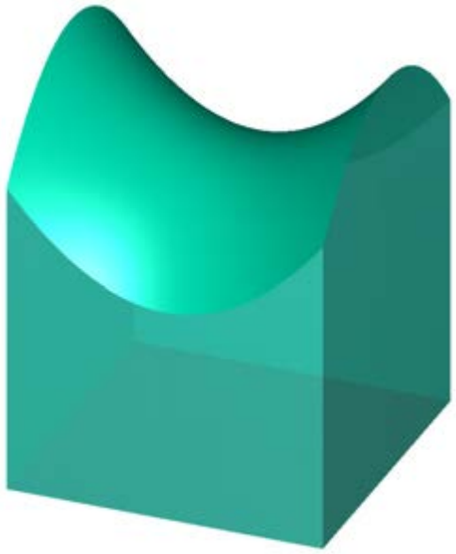

Figure 2.37
Volume
Triple integral of a constant function 1 over the shaded region gives the volume.

A volume integral is a triple integral of the constant function 1, which gives the volume of the region D (*Figure 2.37*). That is, the integral $Vol(D) = \iiint_D dx\,dy\,dz$. It can also mean a triple integral within a region D in R^3 of a function f(x,y,z), and is usually written as:

$$\iiint_D f(x,y,z)\,dx\,dy\,dz.$$

Example: The volume of the cuboid of sides 4 × 6 × 5 may be obtained in two ways:

1. Using the triple integral given above, the volume is equal to $\iiint_{\text{cuboid}} 1\,dx\,dy\,dz$ of the constant function 1 calculated on the cuboid itself. This leads to $\int_{z=0}^{z=5}\int_{y=0}^{z=6}\int_{x=0}^{x=4} 1\,dx\,dy\,dz = 120.$

2. Alternatively, we can use the double integral $\iint_D 5\,dx\,dy$ of the function z = f(x, y) = 5 calculated in the region D in the xy-plane, which is the base of the cuboid. For example, if a rectangular base of such a cuboid is given via the xy inequalities 3 ≤ x ≤ 7, 4 ≤ y ≤ 10, our above double integral now reads $\int_4^{10}\left[\int_3^7 5\,dx\right]dy = 120$. The result will equate to the volume under the surface.

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Average Value of a Function

The average of a function f(x) over an interval [a,b] is $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

KEY POINTS

- An average is a measure of the "middle" or "typical" value of a data set. It is a measure of central tendency.
- If n numbers are given, each number denoted by a_i, where i = 1,..., n, the arithmetic mean is the sum of the a_i's divided by n or $AM = \frac{1}{n}\sum_{i=1}^{n} a_i$.
- An average of a function is equal to the area under the curve, S, divided by the range.

An **average** is a measure of the "middle" or "typical" value of a data set. It is a measure of central tendency. In the most common case, the data set is a discrete set of numbers. The average of a list of numbers is a single number intended to typify the numbers in the list, which is called the **arithmetic mean**. However, the concept of average can be extended to **functions** as well.

If n numbers are given, each number denoted by a_i, where i = 1,..., n, the arithmetic mean is the sum of the a_i's divided by n or $AM = \frac{1}{n}\sum_{i=1}^{n} a_i$. We can extend this definition to continuum by making the following substitution:

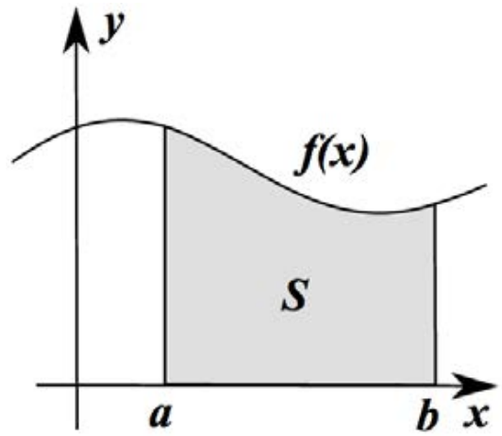

Figure 2.38 Average Value Average of a function f(x) in the range [a,b] is equal to Area/(b-a).

$\sum \to \int, a_i \to f(x), \frac{1}{n} \to \frac{dx}{b-a}$. Therefore, the average of a function f(x) over an interval [a,b] (b>a) (*Figure 2.38*) is expressed as $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$. Note that the average is equal to the area under the curve, S, divided by the range $\frac{S}{b-a}$.

Mean Value Theorem for Integration

The first mean value theorem for integration states, "If G : [a, b] → R is a continuous function and φ is an integrable function that does not change sign on the interval (a, b), then there exists a number x in (a, b) such that $\int_a^b G(t)\varphi(t)\,dt = G(x)\int_a^b \varphi(t)\,dt$." In particular, if φ(t) = 1 for all t in [a, b], then there exists x in (a, b) such that

$\int_a^b G(t)\,dt = G(x)(b-a)$. The value G(x) is the mean value of G(t) on [a, b] as we saw previously.

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Cylindrical Shells

In the shell method, a function is rotated around an axis and modeled by an infinite number of cylindrical shells, all infinitely thin.

KEY POINTS

- The volume of the solid formed by rotating the area between the curves of f(x) and g(x) when integrating perpendicular to the axis of revolution, is $V = 2\pi \int_a^b x\,|f(x) - g(x)|\,dx$.
- The integrand in the integral is nothing but the volume of the infinitely thin cylindrical shell.
- Integration, as an accumulative process, calculates the integrated volume of a "family" of shells (a shell being the outer edge of a hollow cylinder), giving us the total volume.

Shell integration (also called the shell method) is a means of calculating the **volume** of a solid of **revolution** when integrating perpendicular to the axis of revolution (*Figure 2.39*). (When integrating parallel to the axis of revolution, you should use the disk method.) While less intuitive than disk integration, it usually produces simpler integrals. Intuitively speaking, part of the graph of a function is rotated around an axis, and is modeled by an infinite number of cylindrical shells, all infinitely thin.

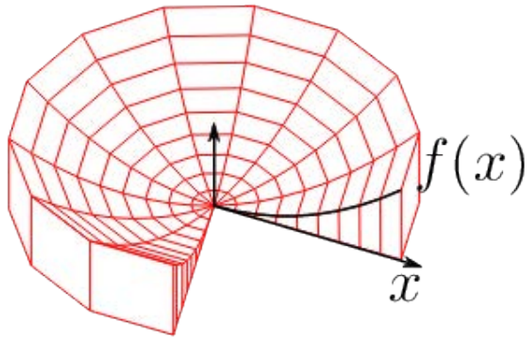

Figure 2.39 The Shell Method

Calculating volume using the shell method. Each segment located at x, between f(x) and x-axis, gives a cylindrical shell after revolution around the vertical axis.

The idea is that a "representative rectangle" (used in the most basic forms of integration – such as ∫ x dx) can be rotated about the axis of revolution, thus generating a hollow **cylinder** with infinitesimal volume. Integration, as an accumulative process, can then calculate the integrated volume of a "family" of shells (a shell being the outer edge of a hollow cylinder), giving us the total volume.

The volume of the solid formed by rotating the area between the curves of f(x) and g(x) and the lines x=a and x=b about the y-axis is given by $V = 2\pi \int_a^b x\,|f(x) - g(x)|\,dx$. In the integrand, the factor "x" represents the radius of the cylindrical shell under consideration, while $|f(x) - g(x)|$ is equal to the height of the shell. Therefore, the entire integrand $2\pi x\,|f(x) - g(x)|\,dx$ is nothing but the volume of the cylindrical shell. By adding the volumes of all these infinitely thin cylinders, we can calculate the volume of the solid formed by the revolution.

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Work

Forces may do work on a system. Work done by a force (F) along a trajectory (C) is given as $\int_C \mathbf{F} \cdot d\mathbf{x}$.

KEY POINTS

- The total work along a path is the time-integral of instantaneous power applied along the trajectory of the point of application: $W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt$.
- The sum of these small amounts of work over the trajectory of the point yields the work:.
- For a constant force directed at an angle θ with the direction of displacement (d), work is given as $W = Fd \cos\theta$.

For moving objects, the rate of the work done by a **force** (measured in joules/second, or watts) is the scalar product of the force (a vector) and the velocity vector of the point of application. This scalar product of force and velocity is classified as instantaneous power delivered by the force. Just as velocities may be integrated over time to obtain a total distance, by the fundamental theorem of calculus, the total work along a path is similarly the time-integral of instantaneous power applied along the trajectory of the point of application.

Work is the result of a force on a point that moves through a distance. As the point moves, it follows a curve X, with a velocity v, at each instant. The small amount of work δW that occurs over an instant of time δt is calculated as $\delta W = \mathbf{F} \cdot \mathbf{v} \delta t$, where the term $\mathbf{F} \cdot \mathbf{v}$ is the power over the instant δt. The sum of these small amounts of work over the trajectory of the point yields the work, $W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{x}}{dt} dt = \int_C \mathbf{F} \cdot d\mathbf{x}$, where C is the trajectory from $x(t_1)$ to $x(t_2)$. This integral is computed along the trajectory of the particle, and is therefore said to be path dependent. If the force is always directed along this line, and the magnitude of the force is F, then this integral simplifies to $W = \int_C F ds$ where s is distance along the line. If F is constant, in addition to being directed along the line, then the integral simplifies further to $W = \int_C F ds = F \int_C ds = Fd$. This calculation can be generalized for a constant force that is not directed along the line, followed by the particle. In this case the dot product $F \cdot dx = F \cos\theta dx$, where θ is the angle between the force vector and the direction of movement, that is $W = \int_C \mathbf{F} \cdot d\mathbf{x} = Fd \cos\theta$.

Example: Work Done by a Spring

Let's consider an object with mass m attached to an ideal spring with spring constant k (*Figure 2.40*). When the object moves from x=x0 to x=0, work done by the spring would be

$$W = \int_C \mathbf{F_s} \cdot d\mathbf{x} = \int_{x_0}^{0} (-kx)dx = \frac{1}{2}kx_0^2.$$

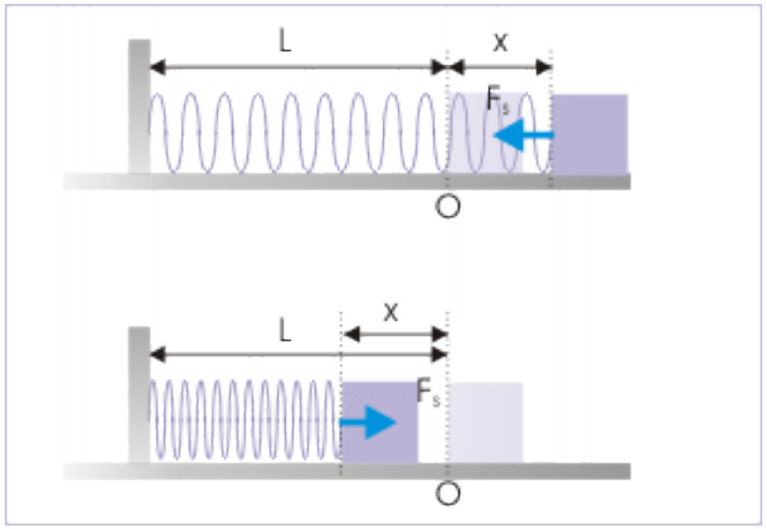

Figure 2.40 Spring and Restoring Force

The spring applies a restoring force (-kx) on the object located at x. Work done by the restoring force leads to increase in the kinetic energy of the object.

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Volumes of Revolution

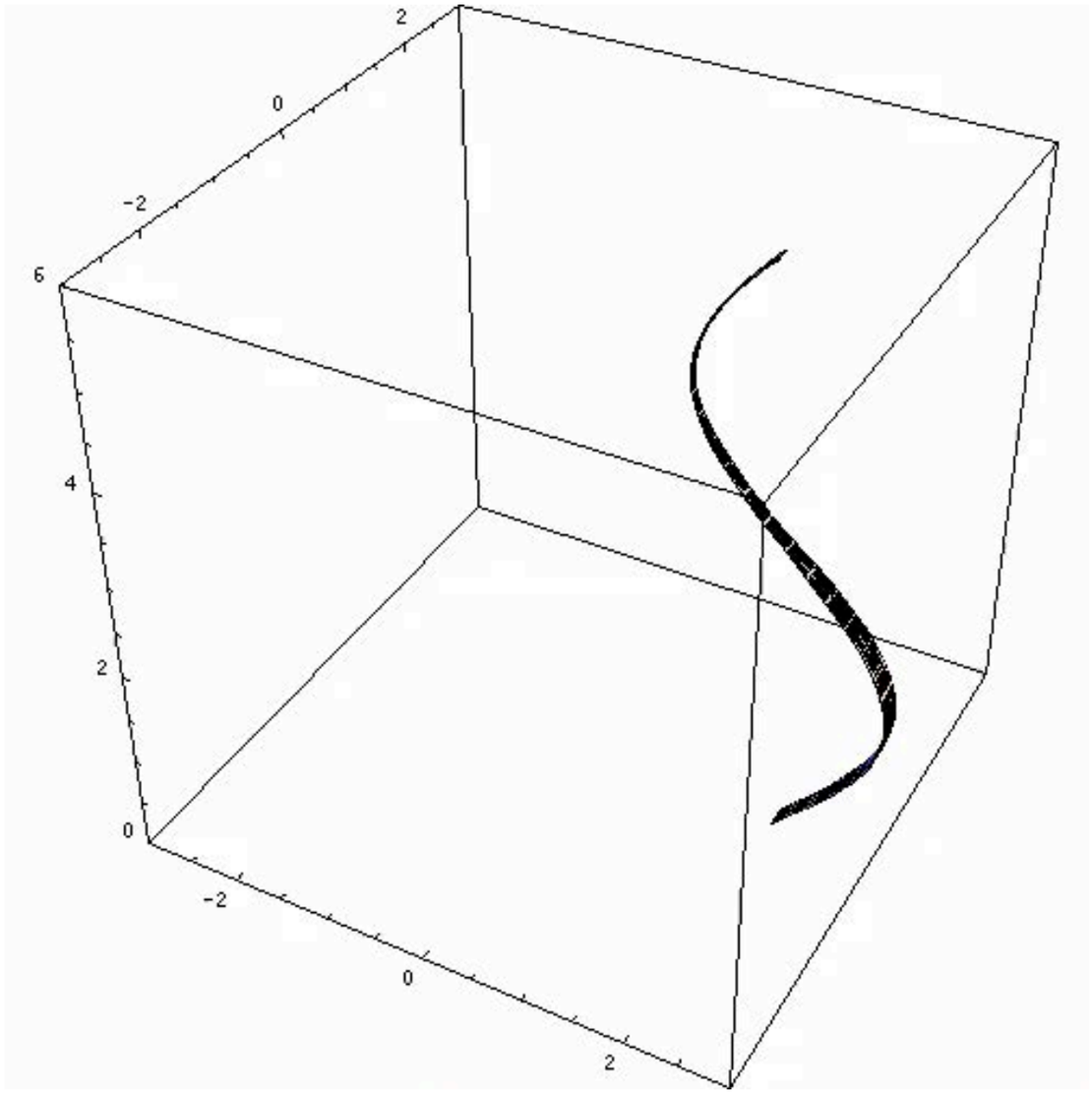
Disc and shell methods of integration can be used to find the volume of a solid produced by revolution.

KEY POINTS

- A solid of revolution is a solid figure obtained by rotating a plane curve around some straight line (the axis) that lies on the same plane.
- The disc method is used when the slice that was drawn is perpendicular to the axis of revolution; i.e. when integrating parallel to the axis of revolution.
- The shell method is used when the slice that was drawn is parallel to the axis of revolution; i.e. when integrating perpendicular to the axis of revolution.

A solid of **revolution** is a solid figure obtained by rotating a plane curve around some straight line (the axis) that lies on the same plane (*Figure 2.41*). Here, we will study how to compute volumes of these objects. Two common methods for finding the volume of a solid of revolution are the disc method and the shell method of **integration**. To apply these methods, it is easiest to draw the graph in question; identify the area that is to be revolved about the axis of revolution; determine the volume of either a disc-shaped slice of the solid, with thickness δx, or a cylindrical shell of width

Figure 2.41 A Volume of Revolution



A solid formed by rotating a curve around an axis.

δx; and then find the limiting sum of these volumes as δx approaches 0, a value which may be found by evaluating a suitable integral.

Disc Method

The disc method is used when the slice that was drawn is perpendicular to the axis of revolution; i.e. when integrating parallel to the axis of revolution (*Figure 2.42*). The volume of the solid formed by rotating the area between the curves of f(x) and g(x) and the lines x=a and x=b about the x-axis is given by $V = \pi \int_a^b |f^2(x) - g^2(x)|\,dx$. If g(x) = 0 (e.g. revolving an area between curve and x-axis), this reduces to: $V = \pi \int_a^b f^2(x)\,dx$ (1).

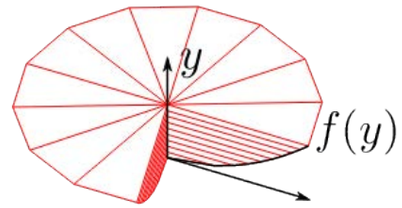


Figure 2.42 Disc Integration

Disc integration about the y-axis. Integration is along the axis of revolution (y-axis in this case).

The method can be visualized by considering a thin horizontal rectangle at y between y=f(x) on top and y=g(x) on the bottom, and revolving it about the y-axis; it forms a ring (or disc in the case that g(x)=0), with outer radius f(x) and inner radius g(x). The area of a ring is $\pi(R^2 - r^2)$, where R is the outer radius (in this case f(x)), and r is the inner radius (in this case g(x)). Summing up all of the areas along the interval gives the total volume. Alternatively, where each disc has a radius of f(x), the discs approach perfect cylinders as their

height dx approaches zero. The volume of each infinitesimal disc is therefore $\pi f^2(x)dx$. An infinite sum of the discs between a and b manifests itself as integral (1).

Shell Method

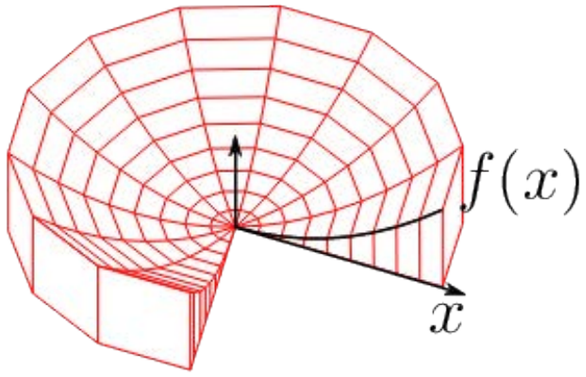


Figure 2.43 Shell Integration

The integration (along the x-axis) is perpendicular to the axis of revolution (y-axis).

The shell method is used when the slice that was drawn is parallel to the axis of revolution; i.e. when integrating perpendicular to the axis of revolution (*Figure 2.43*). The volume of the solid formed by rotating the area between the curves of f(x) and g(x) and the lines x=a and x=b about the y-axis is given by $V = 2\pi \int_a^b x\,|f(x) - g(x)|\,dx$.

If g(x) = 0 (e.g. revolving an area between curve and x-axis), this reduces to: $V = 2\pi \int_a^b x\,|f(x)|\,dx$.

The method can be visualized by considering a thin vertical rectangle at x with height [f(x)-g(x)], and revolving it about the y-axis; it forms a cylindrical shell. The lateral surface area of a cylinder is $2\pi rh$, where r is the radius (in this case x), and h is the height (in this case [f(x)-g(x)]). Summing up all of the surface areas along the interval gives the total volume.

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Chapter 3

Inverse Functions and Advanced Integration

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Section 1

Inverse Functions: Exponential, Logarithmic, and Trigonometric Functions

Inverse Functions

Derivatives of Exponential Functions

Logarithmic Functions

Derivatives of Logarithmic Functions

The Natural Logarithmic Function: Differentiation and Integration

The Natural Exponential Function: Differentiation and Integration

Exponential Growth and Decay

Inverse Trigonometric Functions: Differentiation and Integration

Hyperbolic Functions

Indeterminate Forms and L'Hôpital's Rule

Bases Other than e and their Applications

https://www.boundless.com/calculus/inverse-functions-and-advanced-integration/inverse-functions-exponential-logarithmic-and-trigonometric-functions/

Inverse Functions

An inverse function is a function that undoes another function.

KEY POINTS

- If an input x into the function f produces an output y, then putting y into the inverse function g produces the output x, and vice versa (i.e., f(x)=y, and g(y)=x).
- A function f that has an inverse is called invertible; the inverse function is then uniquely determined by f and is denoted by f^{-1}.
- If f is invertible, the function g is unique; in other words, there is exactly one function g satisfying this property (no more, no less).

An **inverse** function is a **function** that undoes another function. If an input x into the function f produces an output y, then putting y into the inverse function g produces the output x, and vice versa (i.e., f(x)=y, and g(y)=x (*Figure 3.1*). More directly, g(f(x))=x, meaning g(x) composed with f(x), leaves x unchanged. A function f that has an inverse is called invertible; the inverse function is then uniquely determined by f and is denoted by f^{-1}.

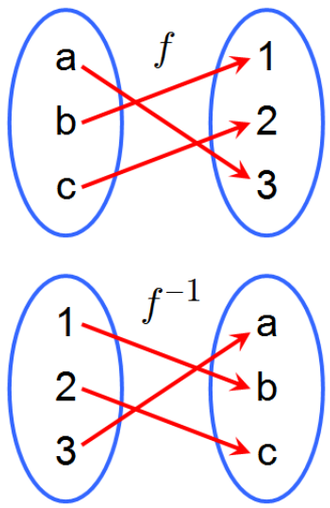

Figure 3.1 A Function and its Inverse

A function f and its inverse, f–1. Because f maps a to 3, the inverse f–1 maps 3 back to a.

Instead of considering the inverses for individual inputs and outputs, one can think of the function as sending the whole set of inputs--the domain-- to a set of outputs--the range. Let f be a function whose domain is the set X and whose range is the set Y. Then f is invertible if there exists a function g with domain Y and range X, with the following property (*Figure 3.2*):

$$f(x) = y \Leftrightarrow g(y) = x$$

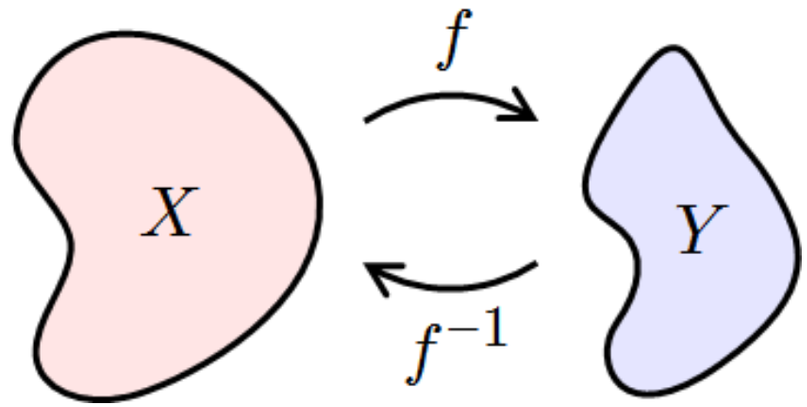

Figure 3.2 Inverse Functions

If f maps X to Y, then f–1 maps Y back to X.

If f is invertible, the function g is unique; in other words, there is exactly one function g satisfying this property (no more, no less). That function g is then called the inverse of f, and is usually denoted as f^{-1}.

Stated otherwise, a function is invertible if and only if its inverse relation is a function on the range Y, in which case the inverse relation is the inverse function. Not all functions have an inverse. For this rule to be applicable, each element $y \in Y$ must correspond to no more than one $x \in X$; a function f with this property is called one-to-one, information-preserving, or an injection.

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Derivatives of Exponential Functions

The derivative of the exponential function is equal to the value of the function.

KEY POINTS

- e^x is its own derivative: $\frac{d}{dx}e^x = e^x$.
- If a variable's growth or decay rate is proportional to its size, then the variable can be written as a constant times an exponential function of time.
- For any differentiable function f(x), $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

The importance of the **exponential** function in mathematics and the sciences stems mainly from properties of its derivative. In particular, $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

That is, e^x is its own derivative, as is shown in (*Figure 3.3*). From any point *P* on the curve (blue), let a tangent line (red), and a vertical line (green) with height *h* be drawn, forming a right triangle with a base *b* on the *x*-axis. Since the slope of the red tangent line (the derivative) at *P* is equal to the ratio of the triangle's height to

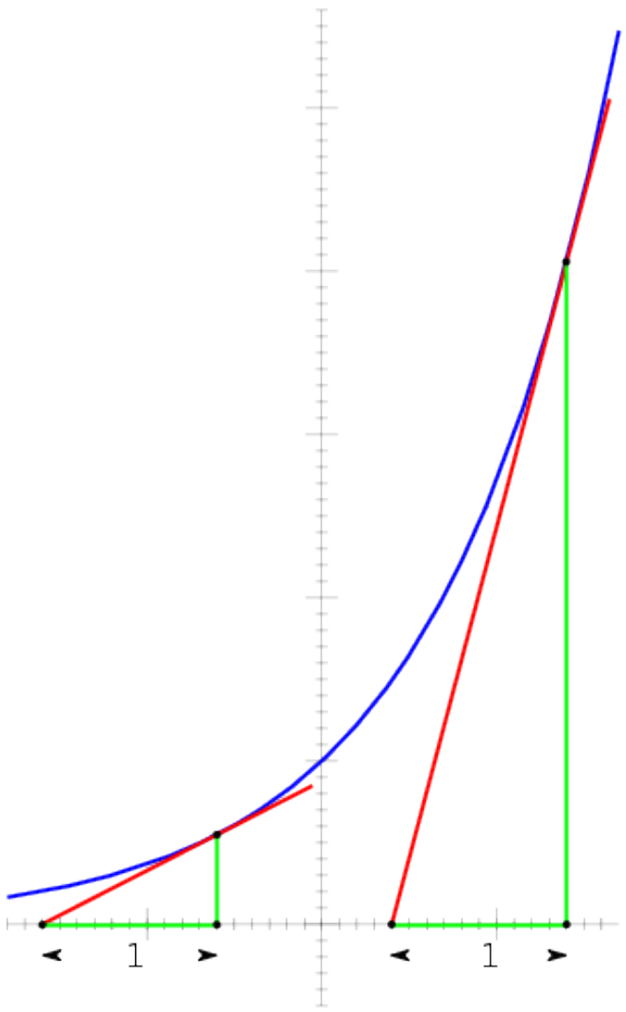

Figure 3.3 Graph of an Exponential Function

Graph of the exponential function illustrating that its derivative is equal to the value of the function.

the triangle's base (rise over run), and the derivative is equal to the value of the function, *h* must be equal to the ratio of *h* to *b*. Therefore, the base *b* must always be 1.

Functions of the form ce^x for constant c are the only functions with that property. Other ways of saying the same thing include a) The slope of the graph at any point is the height of the function at that point; b) The rate of increase of the function at x is equal to the value of the function at x; c) The function solves the differential equation $y' = y$; and d) exp is a fixed point of derivative as a functional.

If a variable's growth or decay rate is proportional to its size—as is the case in unlimited population growth, continuously compounded interest, or radioactive decay—then the variable can be written as a constant times an exponential function of time. Explicitly for any real constant k, a function $f: \mathbb{R} \to \mathbb{R}$ satisfies $f' = kf$ if and only if $f(x) = ce^{kx}$ for some constant c.

Furthermore, for any differentiable function f(x), we find, by the chain rule: $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

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Logarithmic Functions

The logarithm of a number is the exponent by which another fixed value must be raised to produce that number.

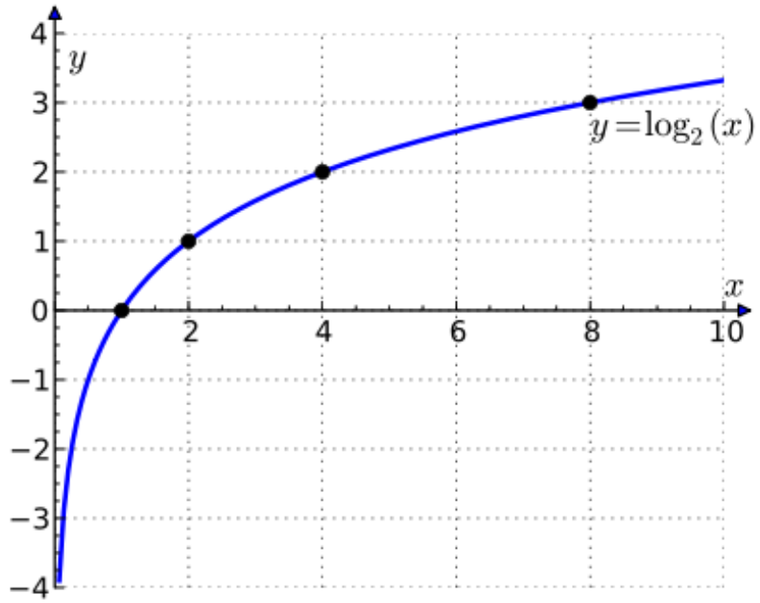
KEY POINTS

- The idea of logarithms is to reverse the operation of exponentiation, that is raising a number to a power.
- A naive way of defining the logarithm of a number x with respect to base b is the exponent by which b must be raised to yield x.
- To define the logarithm, the base b must be a positive real number not equal to 1 and x must be a positive number.

The logarithm of a number is the **exponent** by which another fixed value, the base, must be raised to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 1000 is 10 to the power 3: $1000 = 10 \cdot 10 \cdot 10 = 10^3$. More generally, if $x = b^y$, then y is the logarithm of x to base b, and is written $y = \log_b(x)$, so $\log_{10}(1000) = 3$.The logarithm to base b = 10 is called the common logarithm and has many applications in science and engineering. The natural logarithm has the constant e (≈ 2.718) as its base; its use is widespread in pure mathematics, especially calculus. The **binary** logarithm uses base b = 2 and is prominent in computer science.

The idea of logarithms is to reverse the operation of exponentiation, that is raising a number to a power. For example, the third power (or cube) of 2 is 8, because 8 is the product of three factors of 2: $2^3 = 2 \times 2 \times 2 = 8$. It follows that the logarithm of 8 with respect to base 2 is 3, so $\log_2 8 = 3$ (*Figure 3.4*).

Figure 3.4 Graph of Log Base 2



The graph of the logarithm to base 2 crosses the x axis (horizontal axis) at 1 and passes through the points with coordinates (2, 1), (4, 2), and (8, 3). For example, $log_2(8) = 3$, because $2^3 = 8$. The graph gets arbitrarily close to the y axis, but does not meet or intersect it.

A naive way of defining the logarithm of a number x with respect to base b is the exponent by which b must be raised to yield x. In other

words, the logarithm of x to base b is the solution y to the equation: $b^y = x$.

This definition assumes that we know exactly what we mean by 'raising a real positive number to a real power'. Raising to integer powers is easy. It is clear that two raised to the third is eight, because 2 multiplied by itself 3 times is 8, so the logarithm of eight with respect to base two will be 3.

However, the definition also assumes that we know how to raise numbers to non-integer powers. What would be the logarithm of ten? The definition tells us that the binary logarithm of ten is 3.3219 because two raised to the 3.3219th power is ten. So, the definition only makes sense if we know how to multiply 2 by itself 3.3219 times.

For the definition to work, it must be understood that ' raising two to the 0.3219 power' means 'raising the 10000th root of 2 to the 3219th power'. The tenthousandth root of 2 is 1.0000693171 and this number raised to the 3219th power is 1.2500, therefore ' 2 multiplied by itself 3.3219 times' will be 2 x 2 x 2 x 1.2500 namely 10.

Making this proviso, if the base b is any positive number except 1, and the number x is greater than zero, there is always a real number y that solves the equation: $b^y = x$ so the logarithm is well defined.

The logarithm is denoted "$\log_b(x)$". In the equation $y = \log_b(x)$, the value y is the answer to the question "To what power must b be raised, in order to yield x?". To define the logarithm, the base b must be a positive real number not equal to 1 and x must be a positive number.

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Derivatives of Logarithmic Functions

The general form of the derivative of a logarithmic function is $\frac{d}{dx}log_b(x) = \frac{1}{xln(b)}$.

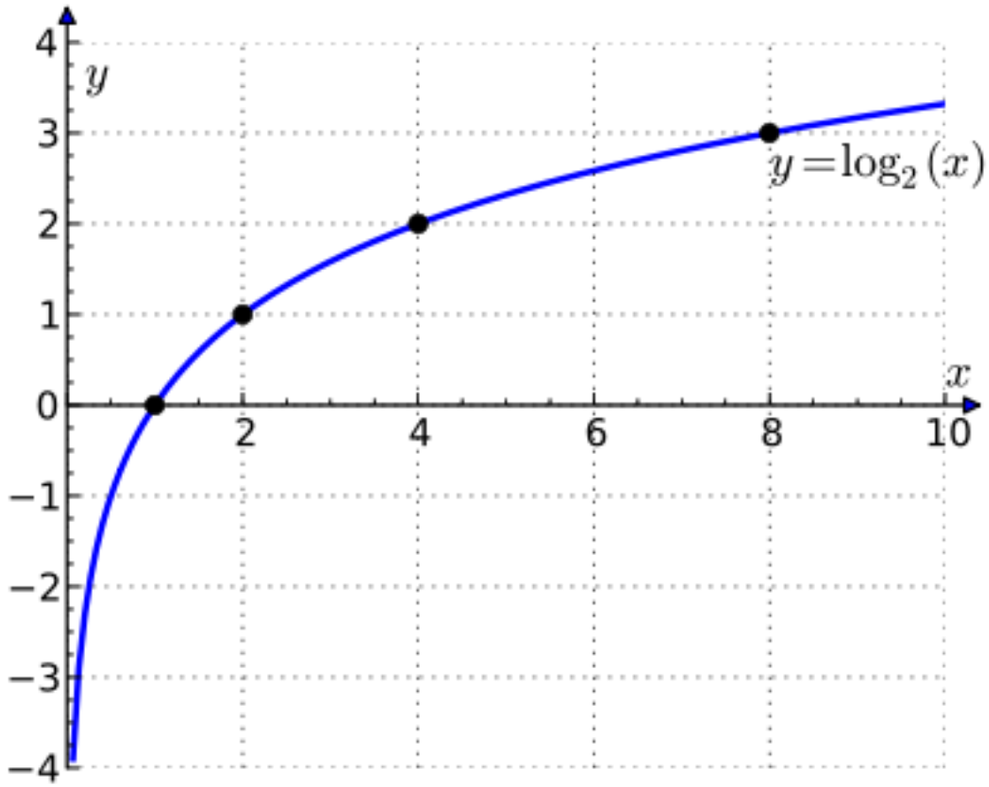
KEY POINTS

- The derivative of natural logarithmic function is $\frac{d}{dx}ln(x) = \frac{1}{x}$.
- The general form of the derivative of a logarithmic function can be derived from the derivative of a natural logarithmic function.
- Properties of the logarithm can be used to to differentiate more difficult functions, such as products with many terms, quotients of composed functions, or functions with variable or function exponents.

Here, we will cover derivatives of logarithmic functions (*Figure 3.5*). First, we will derive the equation for a specific case (the natural log, where the base is **e**), and then we will work to generalize it for any **logarithm**.

Let us create a variable y such that $y = ln(x)$.

Figure 3.5 Graph of the Logarithmic Function



The graph of the logarithm to base 2 crosses the x axis (horizontal axis) at 1 and passes through the points with coordinates (2, 1), (4, 2), and (8, 3). For example, $log_2(8) = 3$, because $2^3 = 8$. The graph gets arbitrarily close to the y axis, but does not meet or intersect it.

It should be noted that what we want to find is the derivative of y, or $\frac{dy}{dx}$.

Next, we will raise both sides to the power of e in an attempt to remove the logarithm from the right hand side: $e^y = x$.

Applying the chain rule and the property of exponents we derived earlier, we can take the derivative of both sides: $\frac{dy}{dx} \times e^y = 1$.

This leaves us with the derivative $\frac{dy}{dx} = \frac{1}{e^y}$.

Substituting back our original equation of $x = e^y$, we find that $\frac{d}{dx} ln(x) = \frac{1}{x}$.

If we wanted, we could go through that same process again for a generalized base, but it is easier just to use properties of logs and realize that $log_b(x) = \frac{ln(x)}{ln(b)}$.

Since $\frac{1}{ln(b)}$ is a constant, we can just take it outside of the derivative

$\frac{d}{dx} log_b(x) = \frac{1}{ln(b)} \times \frac{d}{dx} ln(x)$, which leaves us with the generalized form of $\frac{d}{dx} log_b(x) = \frac{1}{xln(b)}$.

We can use the properties of the logarithm, particularly the natural log, to differentiate more difficult functions, such as products with many terms, quotients of composed functions, or functions with variable or function exponents. We do this by taking the natural logarithm of both sides and re-arranging terms using the following logarithm laws:

$$log(\frac{a}{b}) = log(a) - log(b)$$

$$log(a^n) = nlog(a)$$

$$log(a) + log(b) = log(ab)$$

and then differentiating both sides implicitly, before multiplying through by y.

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The Natural Logarithmic Function: Differentiation and Integration

Differentiation and integration of natural logarithms is based on the property $\frac{d}{dx} ln(x) = \frac{1}{x}$.

KEY POINTS

- The natural logarithm allows simple integration of functions of the form g(x) = f '(x)/f(x).
- The natural logarithm can be integrated using integration by parts: $\int ln(x)dx = xln(x) - x + C$.
- The derivative of the natural logarithm eads to the Taylor series for ln(1 + x) around 0:
  $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots for\ |x| \leq 1$ unless x = -1.

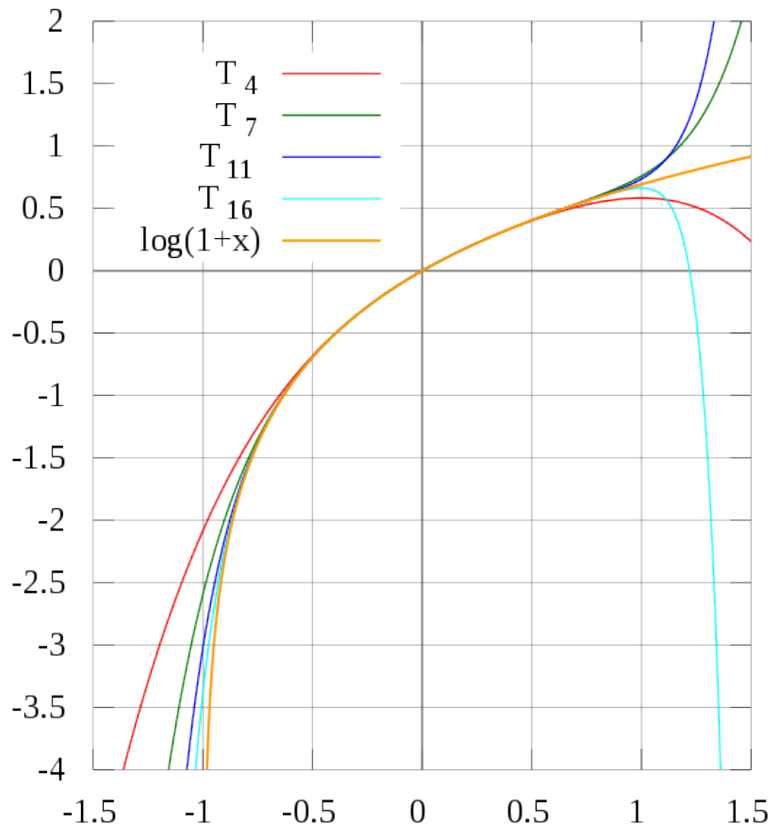
The natural logarithm, generally written as ln x, is the logarithm to the base e, where e is an **irrational** and **transcendental** constant approximately equal to 2.718281828.

The derivative of the natural logarithm is given by $\frac{d}{dx} ln(x) = \frac{1}{x}$.

This leads to the Taylor series for ln(1 + x) around 0:

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots for\ |x| \leq 1, \text{ unless } x = -1.$$

Figure 3.6 Taylor Polynomial for ln(1+x)



The Taylor polynomials for ln(1 + x) only provide accurate approximations in the range −1 < x ≤ 1. Note that, for x > 1, the Taylor polynomials of higher degree are worse approximations.

Figure 3.6 is a picture of ln(1 + x) and some of its Taylor polynomials around 0. These approximations converge to the function only in the region −1 < x ≤ 1; outside of this region the

higher-degree Taylor polynomials are worse approximations for the function.

Substituting x − 1 for x, we obtain an alternative form for ln(x) itself, namely $ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots for \left|x-1\right| \leq 1$ unless $x = 0$.

By using the Euler transform, one obtains the following, which is valid for any x with absolute value greater than 1:

$$ln\frac{x}{x-1} = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \ldots.$$

The natural logarithm allows simple integration of functions of the form g(x) = f '(x)/f(x): an antiderivative of g(x) is given by ln(|f(x)|). This is the case because of the chain rule and the following fact:

$$\frac{d}{dx}(ln(\left|x\right|)) = \frac{1}{x}.$$

In other words, $\int \frac{1}{x}dx = ln\left|x\right| + C$

and $\int \frac{f'(x)}{f(x)}dx = ln\left|f(x)\right| + C.$

Here is an example in the case of g(x) = tan(x):

$$\int tan(x)dx = \int \frac{sin(x)}{cos(x)}dx$$

$$\int tan(x)dx = \int \frac{\frac{-d}{dx}cos(x)}{cos(x)}dx$$

Letting f(x) = cos(x) and f'(x)= − sin(x):

$$\int tan(x)dx = -ln\left|cos(x)\right| + C$$

where C is an arbitrary constant of integration.

The natural logarithm can be integrated using integration by parts:

$$\int ln(x)dx = xln(x) - x + C.$$

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The Natural Exponential Function: Differentiation and Integration

The derivative of the exponential function $\frac{d}{dx}a^x = ln(a)a^x$.

KEY POINTS

- The formula for differentiation of exponential function e^x.
- The derivative of the natural exponential function $\frac{d}{dx}e^x = e^x$.
- The integral of the natural exponential function $\int e^x dx = e^x + C$.

Here we consider **differentiation** of natural exponential functions (*Figure 3.7*). First, we determine the derivative of e^x using the definition of the derivative: $\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$.

Then we apply some basic algebra with powers:

$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$.

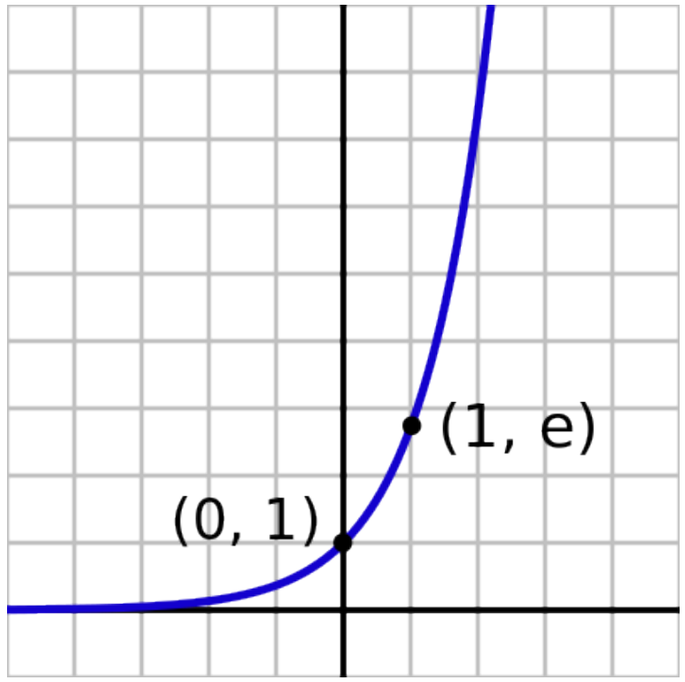

Figure 3.7 Natural Exponential Function

The natural exponential function $y = e^x$

Since e^x does not depend on h, it is constant as h goes to 0. Thus, we can use the limit rules to move it to the outside, leaving us with

$\frac{d}{dx}e^x = e^x \lim_{h\to 0}\frac{e^h - 1}{h}$.

The limit can then be calculated using L'Hôpital's rule:

$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$.

Now we have proved the following rule: $\frac{d}{dx}e^x = e^x$.

Now that we have derived a specific case, let us extend things to the general case of exponential function. Assuming that a is a positive

real constant, we wish to calculate the following: $\frac{d}{dx}a^x$.

Since we have already determined the derivative of e^x, we will attempt to rewrite a^x in that form. Using that $e^{\ln(c)} = c$ and that $\ln(a^b) = b \cdot \ln(a)$, we find that: $a^x = e^{x \times ln(a)}$.

Thus, we simply apply the chain rule:

$$\frac{d}{dx}e^{x \times ln(a)} = \frac{d}{dx}[x \times ln(a)]e^{x \times ln(a)} = ln(a)a^x$$

Derivative of the exponential function: $\frac{d}{dx}a^x = ln(a)a^x$.

Here we consider integration of natural exponential function. Note that the exponential function $y = e^x$ is defined as the inverse of $ln(x)$. Therefore $ln(e^x) = x$ and $e^{lnx} = x$. Let's consider the example of $\int e^x dx$. Since $e^x = (e^x)'$ we can integrate both sides to get

$$\int e^x dx = e^x + C.$$

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Exponential Growth and Decay

Exponential growth occurs when the growth rate of the value of a mathematical function is proportional to the function's current value.]

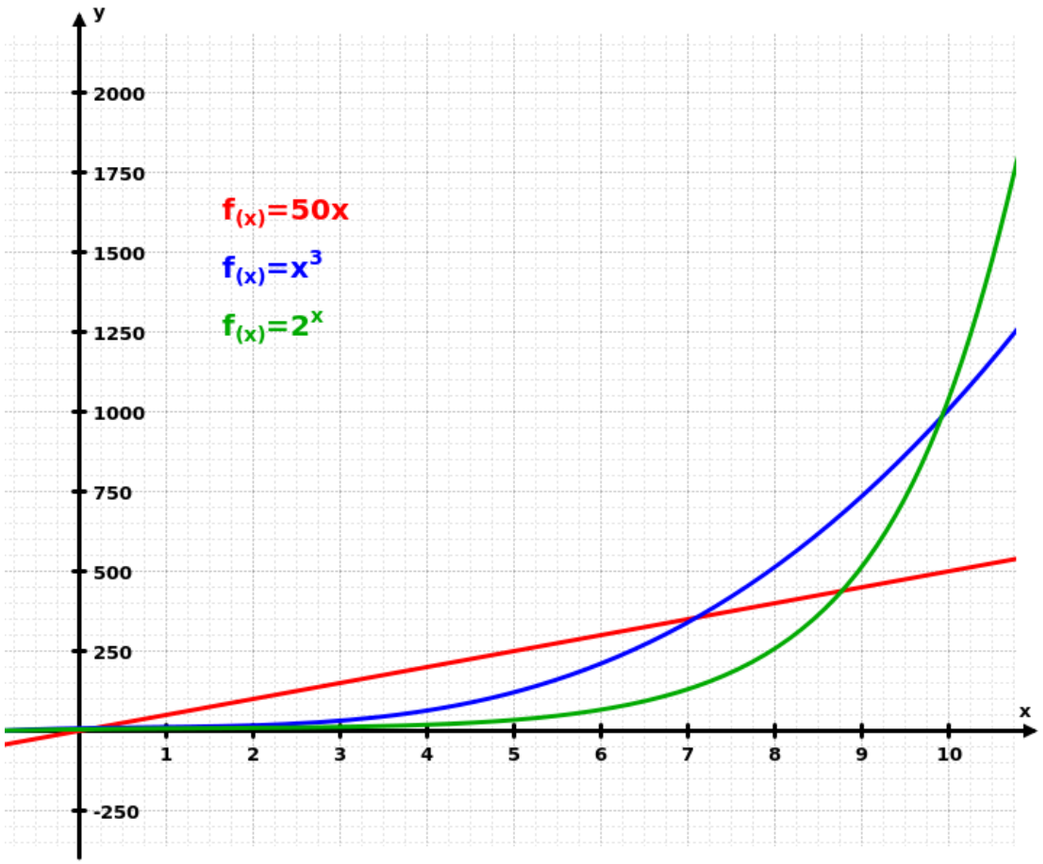
KEY POINTS

- The formula for exponential growth of a variable x at the (positive or negative) growth rate r, as time t goes on in discrete intervals (that is, at integer times 0, 1, 2, 3,...), is: $x_t = x_0(1 + r^t)$ where x_0 is the value of x at time 0.
- Exponential decay occurs in the same way as exponential growth, providing the growth rate is negative.
- In the long run, exponential growth of any kind will overtake linear growth of any kind as well as any polynomial growth.

Exponential growth occurs when the growth rate of the value of a mathematical function is proportional to the function's current value. Exponential decay occurs in the same way, providing the growth rate is negative.

In the long run, exponential growth of any kind will overtake **linear** growth of any kind as well as any **polynomial** growth, as shown in *Figure 3.8*.

Figure 3.8 Exponential Growth



The graph illustrates how exponential growth (green) surpasses both linear (red) and cubic (blue) growth.

The formula for exponential growth of a variable x at the (positive or negative) growth rate r, as time t goes on in discrete intervals (that is, at integer times 0, 1, 2, 3,...), is: $x_t = x_0(1 + r^t)$ where x_0 is the value of x at time 0. For example, with a growth rate of r = 5% = 0.05, going from any integer value of time to the next integer causes x at the second time to be 1.05 times (i.e., 5% larger than) what it was at the previous time.

A quantity x depends exponentially on time t if $x_t = ab^{\frac{t}{\tau}}$ where the constant a is the initial value of x, $x(0) = a$, the constant b is a positive growth factor, and τ is the time constant— the time required for x to increase by one factor of b:
$x(\tau + t) = ab^{\frac{\tau + t}{\tau}} = ab^{\frac{t}{\tau}}b^{\frac{\tau}{\tau}} = x(t)b.$

If $\tau > 0$ and b > 1, then x has exponential growth. If $\tau < 0$ and b > 1, or $\tau > 0$ and $0 < b < 1$, then x has exponential decay.

Let's assume that a species of bacteria doubles every ten minutes. Starting out with only one bacterium, how many bacteria would be present after one hour? The question implies a = 1, b = 2, and τ = 10 min.

$$x(t) = ab^{\frac{t}{\tau}} = 1 \times 2^{\frac{60min}{10min}}$$

$$x(1hour) = 1 \times 2^6 = 64$$

After one hour, or six ten-minute intervals, there would be sixty-four bacteria.

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Inverse Trigonometric Functions: Differentiation and Integration

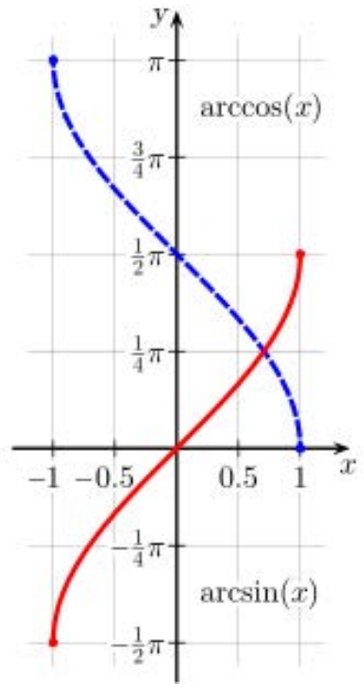
It is useful to know the derivatives and antiderivatives of the inverse trigonometric functions.

KEY POINTS

- The inverse trigonometric functions "undo" the trigonometric functions sin, cos, and tan.
- The inverse trigonometric functions are arcsin, arccos, and arctan.
- Memorizing their derivatives and antiderivatives can be useful.

The inverse **trigonometric** functions are also known as the "arc functions". There are three common notations for inverse trigonometric functions. The arcsine function, for instance, could be written as sin^−1, asin, or, as is used on this page, arcsin. Inverse trigonometric functions include arcsin, arccos (*Figure 3.9*); arctan and arccot (*Figure 3.10*); and arcsec and arccsc (*Figure 3.11*). They can be thought of as the inverse of the corresponding trigonometric functions.

Figure 3.9 Inverse trig functions



The usual principal values of the arcsin(x) (red) and arccos(x) (blue) functions graphed on the cartesian plane.

The differentiation of trigonometric functions is the mathematical process of finding the rate at which a trigonometric function changes with respect to a variable.

The integration of trigonometric functions involves finding the antiderivative.

The following is a list of indefinite integrals (antiderivatives) of expressions involving the inverse trigonometric functions.

$$\int \arcsin(ax)\,dx = x\arcsin(ax) + \frac{\sqrt{1-a^2x^2}}{a} + C$$

$$\int \arccos(ax)\,dx = x\arccos(ax) - \frac{\sqrt{1-a^2x^2}}{a} + C$$

$$\int \arctan(ax)\,dx = x\arctan(ax) - \frac{\ln(a^2x^2+1)}{2a} + C$$

$$\int arccot(ax)dx = x\ arccot(ax) + \frac{\ln(a^2x^2+1}{2a} + C$$

$$\int arcsec(ax)dx = x\ arcsec(ax) - \frac{1}{a}arctanh(\sqrt{1-\frac{1}{a^2x^2}} + C$$

$$\int arccsc(ax)dx = x\ arccsc(ax) + \frac{1}{a}arctanh(\sqrt{1-\frac{1}{a^2x^2}} + C$$

C is used for the arbitrary constant of integration that can only be determined if something about the value of the integral at some point is known. Thus each function has an infinite number of antiderivatives.

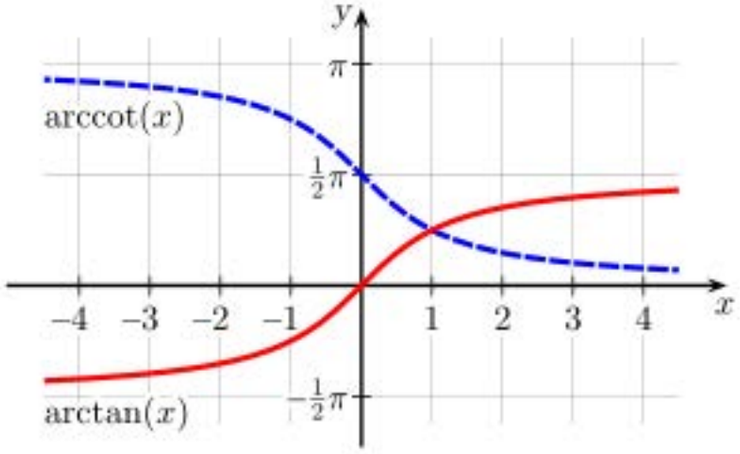


Figure 3.10 Inverse trigonometric functions

The usual principal values of the arctan(x) and arccot(x) functions graphed on the cartesian plane

The derivatives of the inverse trigonometric functions are as follows:

$$d/dx\ arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$d/dx\ arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$d/dx\ arctan(x) = \frac{1}{1+x^2}$$

$$d/dx\ arccot(x) = \frac{-1}{1+x^2}$$

$$d/dx\ arcsec(x) = \frac{1}{x^2\sqrt{1-x^{-2}}}$$

$$d/dx\ arccsc(x) = \frac{-1}{x^2\sqrt{1-x^{-2}}}$$

Note that some of these functions are not valid for a range of x which would end up making the function undefined.

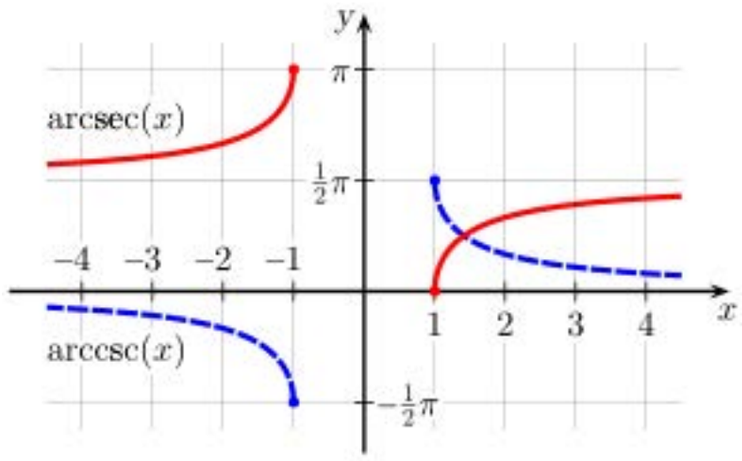


Figure 3.11 Inverse Trigonometric Functions

Principal values of the arcsec(x) and arccsc(x) functions graphed on the cartesian plane.

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Hyperbolic Functions

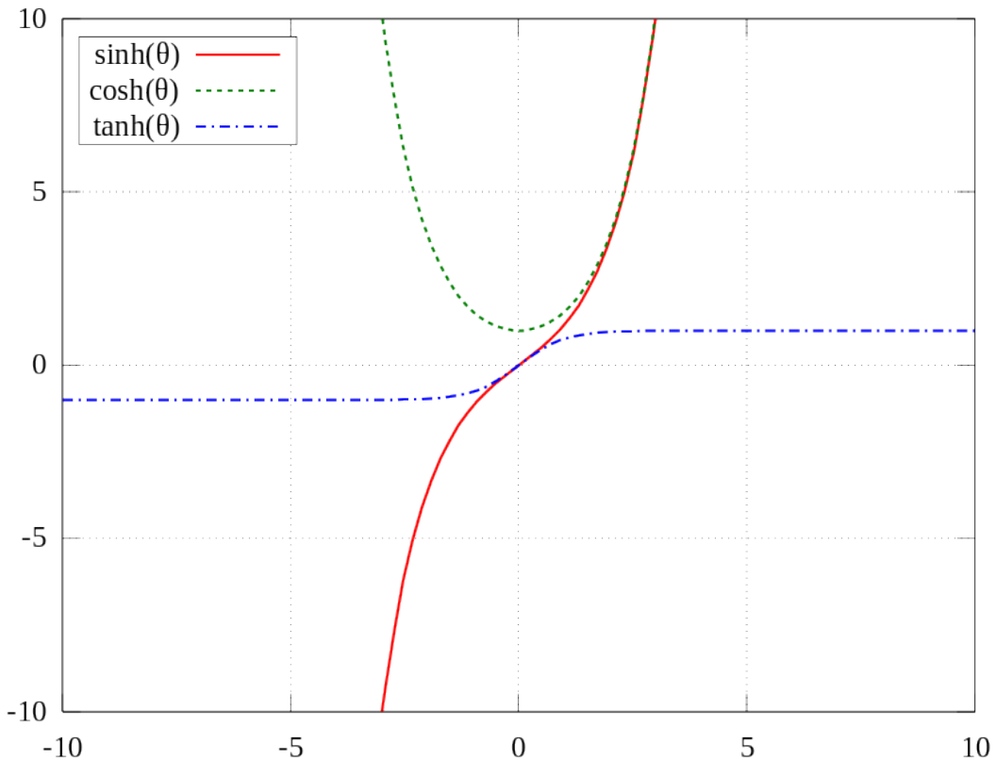
sinh and cosh are basic hyperbolic functions; sinh is defined as the following: $sinh(x) = \frac{e^x - e^{-x}}{2}$.

KEY POINTS

- The basic hyperbolic functions are the hyperbolic sine "sinh", and the hyperbolic cosine "cosh", from which are derived the hyperbolic tangent "tanh", and so on, corresponding to the derived trigonometric functions.
- The inverse hyperbolic functions are the area hyperbolic sine "arsinh" (also called "asinh" or sometimes "arcsinh") and so on.
- The hyperbolic functions take real values for a real argument called a hyperbolic angle. The size of a hyperbolic angle is the area of its hyperbolic sector.

Hyperbolic function is an analog of the ordinary trigonometric function, also called circular function. The basic hyperbolic functions are the hyperbolic sine "sinh", and the hyperbolic cosine "cosh", from which are derived the hyperbolic tangent "tanh", and so on, corresponding to the derived functions (*Figure 3.12*). The **inverse** hyperbolic functions are the area hyperbolic sine "arsinh" (also called "asinh" or sometimes "arcsinh") and so on.

Figure 3.12 Hyperbolic Functions



Basic (sinh x, cosh x) and derived (tanh x) hyperbolic functions

Just as the points (cos t, sin t) form a circle with a unit radius, the points (cosh t, sinh t) form the right half of the equilateral hyperbola. Hyperbolic functions occur in the solutions of some important linear differential equations, for example the equation defining a catenary, of some cubic equations, and of Laplace's equation in Cartesian coordinates. The latter is important in many areas of physics, including electromagnetic theory, heat transfer, fluid dynamics, and special relativity.

The hyperbolic functions take real values for a real argument called a hyperbolic angle. The size of a hyperbolic angle is the area of its hyperbolic sector. The hyperbolic functions may be defined in terms of the legs of a right triangle covering this sector.

In complex analysis, the hyperbolic functions arise as the imaginary parts of sine and cosine. When considered defined by a complex variable, the hyperbolic functions are rational functions of exponentials, and are hence **meromorphic**.

The hyperbolic functions are:

Hyperbolic sine: $sinh(x) = \frac{e^x - e^{-x}}{2}$

Hyperbolic cosine: $cosh(x) = \frac{e^x + e^{-x}}{2}$

Hyperbolic tangent: $tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$

Hyperbolic cotangent: $coth(x) = \frac{cosh(x)}{sinh(x)} = \frac{1 + e^{-2x}}{1 - e^{-2x}}$

Hyperbolic secant: $sech(x) = (cosh(x))^{-1} = \frac{2}{e^x + e^{-x}}$

Hyperbolic cosecant: $csch(x) = (sinh(x))^{-1} = \frac{2}{e^x - e^{-x}}$

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Indeterminate Forms and L'Hôpital's Rule

Indeterminate forms like 0/0 have no definite value; however, when a limit is indeterminate, l'Hôpital's rule can often be used to evaluate it.

KEY POINTS

- Indeterminate forms include 0^0, 0/0, 1^∞, ∞ – ∞, ∞/∞, 0 × ∞, and ∞^0.
- Indeterminate forms often arise when you are asked to take the limit of a function. For example: $\lim_{x\to 0} \frac{x}{x}$ is indeterminate, giving 0/0.
- l'Hôpital's rule: For f and g which are differentiable, if $ $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x in the interval containing c, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

Occasionally in mathematics, one runs across an equation with an **indeterminate** form. In calculus and other branches of mathematical analysis, an indeterminate form is an algebraic expression obtained in the context of **limit**s. Limits involving algebraic operations are often performed by replacing

subexpressions by their limits; if the expression obtained after this substitution does not give enough information to determine the original limit, it is known as an indeterminate form. The indeterminate forms include 0^0, 0/0, 1^∞, $\infty - \infty$, ∞/∞, $0 \times \infty$, and ∞^0.

The most common example of an indeterminate form is 0/0. As x approaches 0, the ratios x/x3, x/x, and x2/x go to ∞, 1, and 0, respectively. In each case, however, if the limits of the numerator and denominator are evaluated and plugged into the division operation, the resulting expression is 0/0. So, roughly speaking, 0/0 can be 0, or ∞, or it can be 1 and, in fact, it is possible to construct similar examples converging to any particular value. That is why the expression 0/0 is indeterminate.

More formally, the fact that the functions f and g both approach 0 as x approaches some limit point c is not enough information to evaluate the limit $\lim_{x \to c} \frac{f(x)}{g(x)}$. That limit could converge to any number, or diverge to infinity, or might not exist, depending on what the functions f and g are.

For example, $\lim_{x \to 0} \frac{x}{x}$ is indeterminate.

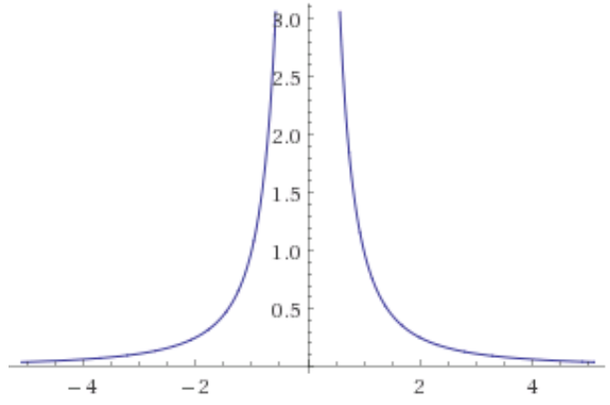

Figure 3.13
Example of Indeterminancy
As x goes to 0 in the graph of x/x^3, the value approaches 0/0, which is indeterminate.

L'Hôpital's Rule

In calculus, l'Hôpital's rule uses derivatives to help evaluate limits involving indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to a determinate form, allowing easy evaluation of the limit.

In its simplest form, l'Hôpital's rule states that for functions f and g which are differentiable, if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x in the interval containing c, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

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Bases Other than e and their Applications

Among all choices for the base (b), b = e, 2, and 10 are particularly common for logarithms.

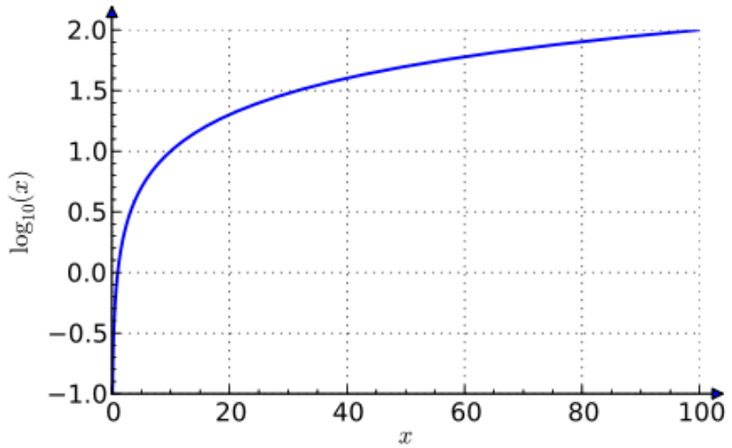
KEY POINTS

- The major advantage of common logarithms (logarithms to base ten) is that they are easy to use for manual calculations in the decimal number system.
- The binary logarithm is often used in computer science and information theory because it is closely connected to the binary numeral system.
- Common logarithm is frequently written as "log(x)"; binary logarithm is frequently written "ld n" or "lg n".

Among all choices for the base b, three are particularly common for **logarithms**. These are b = 10 (common logarithm; see *Figure 3.14*, b = e (natural logarithm), and b = 2 (binary logarithm; see *Figure 3.15*. In this atom we will focus on common and binary logarithms.

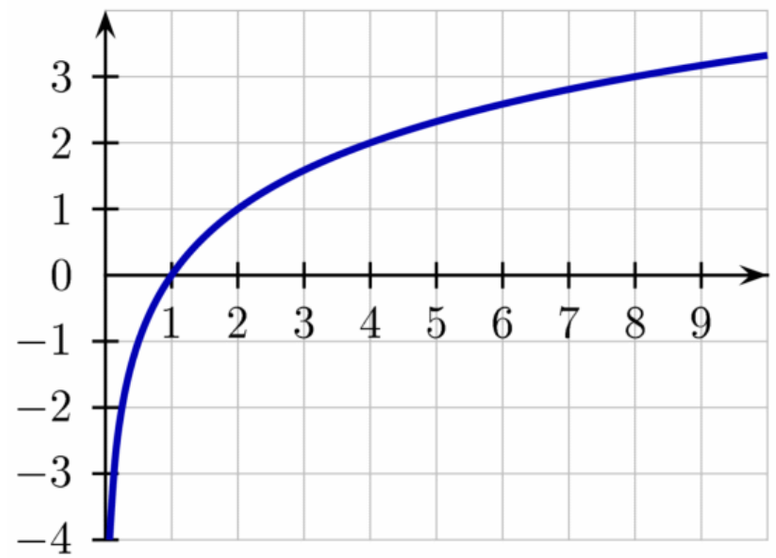
The major advantage of common logarithms (logarithms in base ten) is that they are easy to use for manual calculations in the decimal number system:

Figure 3.14 Logarithmic Function



Graph of a common logarithm function.

Figure 3.15 Binary Logarithmic Function



Graph of a binary logarithm function.

$log_{10}(10x) = log_{10}(10) + log_{10}(x) = 1 + log_{10}(x)$. Thus, $\log_{10}(x)$ is related to the number of decimal digits of a positive integer x: the number of digits is the smallest integer strictly bigger than $\log_{10}(x)$.

For example, $\log_{10}(1430)$ is approximately 3.15. The next integer is 4, which is the number of digits of 1430.

Before the early 1970s, hand-held electronic calculators were not yet in widespread use. Due to their utility in saving work in laborious multiplications and divisions with pen and paper, tables of base 10 logarithms were given in appendices of many books. Such a table of "common logarithms" gave the logarithm--often to 4 or 5 decimal places-- of each number in the left-hand column, which ran from 1 to 10 by small increments, perhaps 0.01 or 0.001. There was only a need to include numbers between 1 and 10, since the logarithms of larger numbers were easily calculated.

Because base 10 logarithms were most useful for computations, engineers generally wrote "log(x)" when they meant $\log_{10}(x)$. Mathematicians, on the other hand, wrote "log(x)" when they meant log_e(x) for the natural logarithm. Today, both notations are found. Since hand-held electronic calculators are designed by engineers rather than mathematicians, they customarily follow engineers' notation.

Binary logarithm ($\log_2 n$) is the logarithm in base 2. It is the inverse function of $n \Rightarrow 2^n$. The binary logarithm of n is the power to which the number 2 must be raised to obtain the value n. This makes the binary logarithm useful for anything involving powers of 2 (i.e., doubling). For example, the binary logarithm of 1 is 0, the binary logarithm of 2 is 1, the binary logarithm of 4 is 2, the binary logarithm of 8 is 3, the binary logarithm of 16 is 4, and the binary logarithm of 32 is 5.

The binary logarithm is often used in computer science and information theory because it is closely connected to the binary numeral system. It is frequently written as "ld n" or "lg n".

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Section 2

Techniques of Integration

Basic Integration Principles

Integration By Parts

Trigonometric Integrals

Trigonometric Substitution

The Method of Partial Fractions

Integration Using Tables and Computers

Approximate Integration

Improper Integrals

Numerical Integration

https://www.boundless.com/calculus/inverse-functions-and-advanced-integration/techniques-of-integration/

Basic Integration Principles

Integration is the process of finding the region bounded by a function; this process makes use of several important properties.

KEY POINTS

- The term integral may also refer to the notion of the anti-derivative, a function F whose derivative is the given function f. In this case, it is called an indefinite integral and is written, $F = \int f(x)\,dx$.
- Integration is linear, additive, and preserves inequality of functions.
- The definite integral of f over the interval a to b is given by $\int_a^b f = F|_a^b$, where F is the anti-derivative of f.

Integration is an important concept in mathematics and--together with its inverse, differentiation-- is one of the two main operations in calculus. Given a function f of a real variable x, and an interval [a, b] of the real line, the definite integral $\int_a^b f(x)\,dx$ is defined informally to be the area of the region in the xy-plane bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b, such that area above the x-axis adds to the total, and that below the x-axis subtracts from the total (*Figure 3.16*). The term integral may also refer to the notion of the anti-derivative, a function F whose derivative is the given function f. In this case, it is called an indefinite integral and is written, $F = \int f(x)\,dx$.

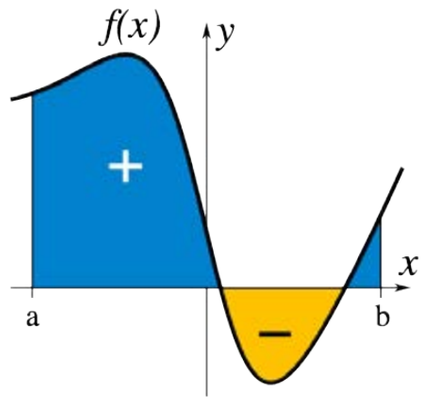

Figure 3.16
Definite Integral
A definite integral of a function can be represented as the signed area of the region bounded by its graph.

More rigorously, once an anti-derivative F of f is known for a continuous real-valued function f defined on a closed interval [a, b], the definite integral of f over that interval is given by

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Integration proceeds by adding up an infinite number of infinitely small areas. This sum can be computed by using the anti-derivative.

Properties

1. Linearity: The integral of a linear combination is the linear combination of the integrals.

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

2. Inequalities: If f(x) ≤ g(x) for each x in [a, b], then each of the upper and lower sums of f is bounded above by the upper and lower sums, respectively, of g. $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

3. Additivity: If c is any element of [a, b], then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

4. Reversing limits of integration: If a > b,
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

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Integration By Parts

Integration by parts is a way of integrating complex functions by breaking them down into separate parts and integrating them individually.

KEY POINTS

- Integration by parts is a theorem that relates the integral of a product of functions to the integral of their derivative and anti-derivative.
- The theorem is expressed as the following:
$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$
- Integration by parts may be interpreted graphically in addition to mathematically.

Introduction

In calculus, integration by parts is a theorem that relates the integral of a product of functions to the integral of their **derivative** and anti-derivative. It is frequently used to find the anti-derivative of a product of functions into an ideally simpler anti-derivative. The rule can be derived in one line by simply integrating the product rule of differentiation.

Theorem of integration by parts

Let's take the functions u = u(x) and v = v(x). When taking their derivatives, we are left with du = u '(x) and dxdv = v'(x) dx. Now, let's take a look at the principal of integration by or, more compactly, $\int u\,dv = uv - \int v\,du$

Proof

Suppose u(x) and v(x) are two continuously differentiable functions. The product rule states:

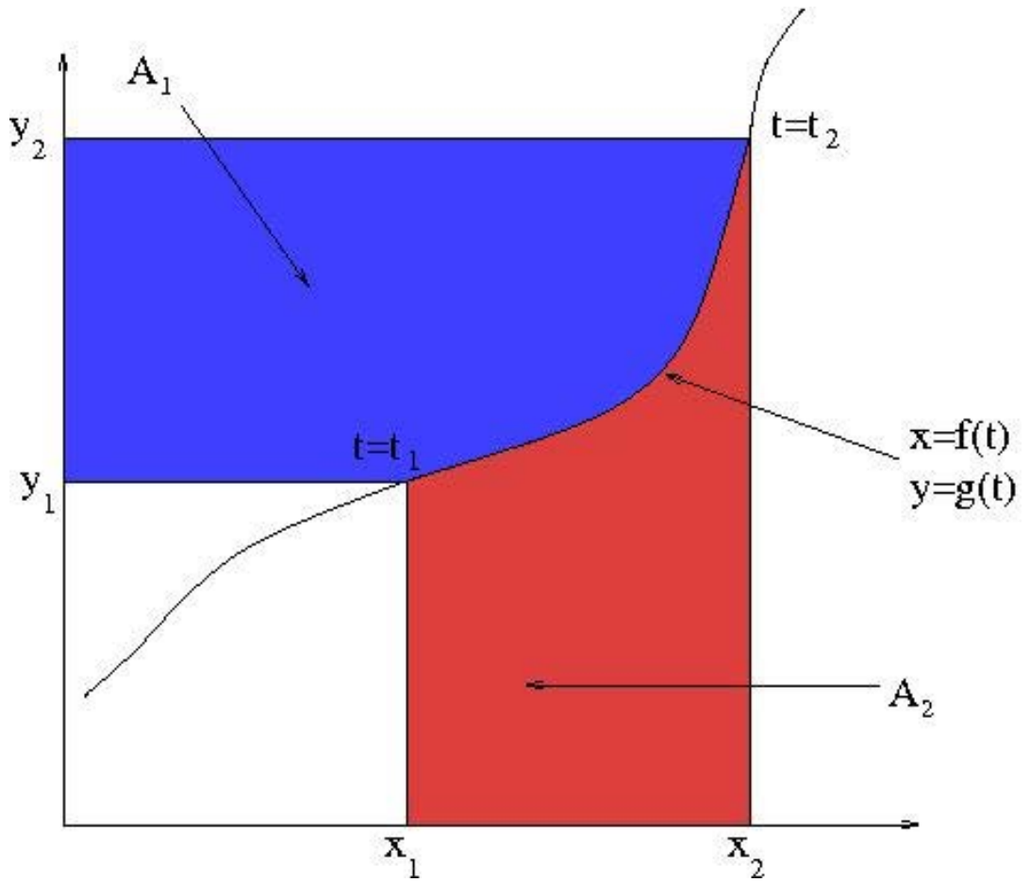
$$\frac{d}{dx}\left(u(x)v(x)\right) = u(x)\frac{d}{dx}\left(v(x)\right) + \frac{d}{dx}\left(u(x)\right)v(x)$$

Integrating both sides with respect to x, over an interval a ≤ x ≤ b, $\int_a^b \frac{d}{dx}\left(u(x)v(x)\right)\,dx = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx$, then applying the fundamental theorem of calculus, $\int_a^b \frac{d}{dx}\left(u(x)v(x)\right)\,dx = \left[u(x)v(x)\right]_a^b$ gives the formula for "integration by parts": $\left[u(x)v(x)\right]_a^b = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx.$

Visualization

Let's define a parametric curve by (x, y) = (f(t), g(t)) (*Figure 3.17*).

Figure 3.17 Integration By Parts



Integration by parts may be thought of as deriving the area of the blue region from the total area and that of the red region.

The area of the blue region is $A_1 = \int_{y_1}^{y_2} x(y)dy$. Similarly, the area of the red region is $A_2 = \int_{x_1}^{x_2} y(x)dx$. The total area, A_1+A_2, is equal to the area of the bigger rectangle, x_2y_2, minus the area of the smaller one,

$$x_1y_1: \overbrace{\int_{y_1}^{y_2} x(y)dy}^{A_1} + \overbrace{\int_{x_1}^{x_2} y(x)dx}^{A_2} = x_i y_i \Big|_{i=1}^{i=2}$$

Assuming the curve is smooth within a neighborhood, this generalizes to indefinite integrals $\int x\,dy + \int y\,dx = xy$, which can be rearranged to the form of the theorem $\int x\,dy = xy - \int y\,dx$

Example

In order to calculate $I = \int x\cos(x)\,dx$, let

$u = x \Rightarrow du = dx$

$dv = \cos(x)\,dx \Rightarrow v = \int \cos(x)\,dx = \sin x$

then

$\int x\cos(x)\,dx$

$= \int u\,dv$

$= uv - \int v\,du$

$= x\sin(x) - \int \sin(x)\,dx$

$= x\sin(x) + \cos(x) + C.$

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Trigonometric Integrals

The trigonometric integrals are a specific set of functions used to simplify complex mathematical expressions in order to evaluate them.

KEY POINTS

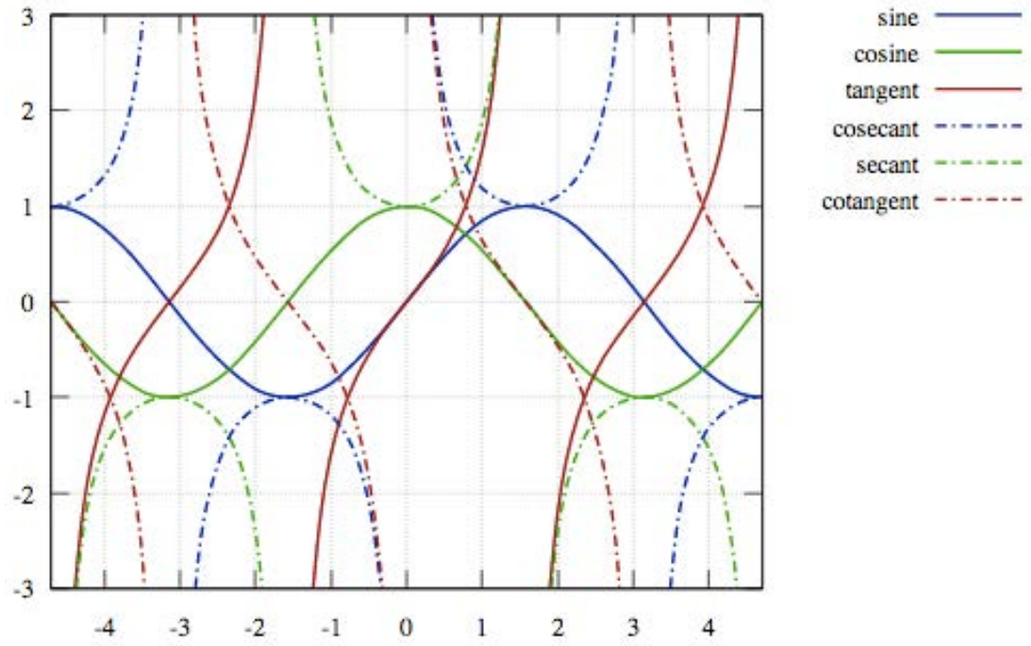
- Some of the expressions for the trigonometric integrals are found using properties of trigonometric functions.
- Some of the expressions were derived using techniques like integration by parts.
- There is no guarantee that a trigonometric integral has an analytic expression.

Trigonometric Integrals

The **trigonometric** integrals are a family of integrals which involve trigonometric functions (sin, cos, tan, etc...) (*Figure 3.18*). The following is a list of integrals of trigonometric functions. Some of them were computed using properties of the trigonometric functions, while others used techniques such as integration by parts. Generally, if the function, sin(x), is any trigonometric function, and cos(x) is its derivative, then

$\int a \cos nx \, dx = \frac{a}{n} \sin nx + C$. In all formulas, the constant a is assumed to be nonzero, while C denotes the integration constant.

Figure 3.18 Graphs of Trigonometry Functions



Trigonometric functions: Sine, Cosine, Tangent, Cosecant (dotted), Secant (dotted), Cotangent (dotted)

Integrals Involving Only Sine

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax + C = \frac{x}{2} - \frac{1}{2a} \sin ax \cos ax + C$$

$$\int \sin^3 ax \, dx = \frac{\cos 3ax}{12a} - \frac{3\cos ax}{4a} + C$$

$$\int x \sin^2 ax \, dx = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax + C$$

$$\int x^2 \sin^2 ax \, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x}{4a^2}\cos 2ax + C$$

Integrands Involving Only Cosine

$$\int \cos ax \, dx = \frac{1}{a}\sin ax + C$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax + C = \frac{x}{2} + \frac{1}{2a}\sin ax \cos ax + C$$

$$\int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax \, dx \qquad (\text{for } n > 0)$$

$$\int x \cos ax \, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a} + C$$

$$\int x^2 \cos^2 ax \, dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x}{4a^2}\cos 2ax + C$$

Integrands Involving Only Tangent

$$\int \tan ax \, dx = -\frac{1}{a}\ln|\cos ax| + C = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \tan^n ax \, dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax \, dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{q\tan ax + p} = \frac{1}{p^2+q^2}(px + \frac{q}{a}\ln|q\sin ax + p\cos ax|) + C \qquad (\text{for } p^2 + q^2 \neq 0)$$

Integrands Involving Only Secant

$$\int \sec ax \, dx = \frac{1}{a}\ln\left|\sec ax + \tan ax\right| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

Integrands Involving Only Cosecant

$$\int \csc ax \, dx = \frac{1}{a}\ln\left|\csc ax - \cot ax\right| + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

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Trigonometric Substitution

Trigonometric functions can be substituted for other expressions to change the form of integrands and simplify the integration.

KEY POINTS

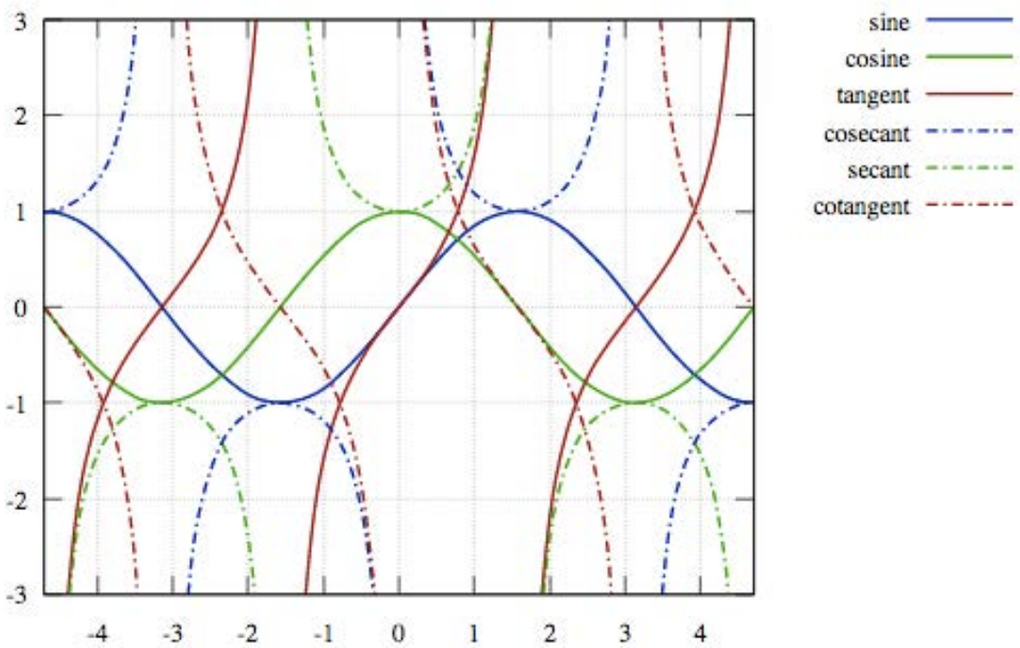
- If the integrand contains $a^2 - x^2$, let $x = a\sin(\theta)$.
- If the integrand contains $a^2 + x^2$, let $x = a\tan(\theta)$ . .
- If the integrand contains $x^2 - a^2$, let $x = a\sec(\theta)$.

Trigonometric functions can be substituted (*Figure 3.19*) for other expressions to change the form of integrands. One may use the trigonometric identities to simplify certain integrals containing radical expressions (or expressions containing n-th roots). Here are general methods of trigonometric substitution, depending on the form of the function to be integrated:

Substitution Rule #1: If the integral contains $a^2 - x^2$, let $x = a\sin(\theta)$ and use the identity: $1 - \sin^2(\theta) = \cos^2(\theta)$.

Substitution Rule #2: If the integrand contains $a^2 + x^2$, let $x = a\tan(\theta)$ and use the identity: $1 + \tan^2(\theta) = \sec^2(\theta)$.

Figure 3.19 Graphs of Trigonometric Functions



Trigonometric functions: Sine, Cosine, Tangent, Cosecant (dotted), Secant (dotted), Cotangent (dotted). Each of these functions can be used in trigonometric substitution to simplify the function for integration and evaluation.

Substitution Rule #3: If the integrand contains $x^2 - a^2$, let $x = a\sec(\theta)$ and use the identity: $\sec^2(\theta) - 1 = \tan^2(\theta)$.

Note that, for a definite integral, one must figure out how the bounds of integration change due to the substitution.

Examples

In order to better understand these substitutions, let's go over the derivation of some of them.

1. Integrals where the integrand contains $a^2 - x^2$ (where a is positive): In the integral $\int \frac{dx}{\sqrt{a^2 - x^2}}$ we may use $x = a\sin(\theta), \quad dx = a\cos(\theta)\,d\theta, \quad \theta = \arcsin\left(\frac{x}{a}\right)$. With the substitution, we get

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2 - a^2\sin^2(\theta)}}$$

$$= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2(1 - \sin^2(\theta))}}$$

$$= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2\cos^2(\theta)}}$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x}{a}\right) + C.$$

2. Integrals where the integrand contains $a^2 - x^2$ (where a is not zero): In the integral $\int \frac{dx}{a^2 + x^2}$, we may use

$x = a\tan(\theta), \quad dx = a\sec^2(\theta)\,d\theta, \quad \theta = \arctan\left(\frac{x}{a}\right)$. With the substitution, we get

$$\int \frac{dx}{a^2 + x^2} = \int \frac{a\sec^2(\theta)\,d\theta}{a^2 + a^2\tan^2(\theta)}$$

$$= \int \frac{a\sec^2(\theta)\,d\theta}{a^2(1 + \tan^2(\theta))}$$

$$= \int \frac{a\sec^2(\theta)\,d\theta}{a^2\sec^2(\theta)}$$

$$= \int \frac{d\theta}{a}$$

$$= \frac{\theta}{a} + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

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The Method of Partial Fractions

Partial fraction expansions provide an approach to integrating a general rational function.

KEY POINTS

- Any rational function of a real variable can be written as the sum of a polynomial and a finite number of rational fractions whose denominator is the power of an irreducible polynomial and whose numerator has a degree lower than the degree of this irreducible polynomial.
- The substitution u = ax + b, du = a dx reduces the integral as the following: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\left|ax+b\right| + C$.
- When there is an irreducible 2nd-degree polynomial in the denominator, complete the square and change the variable.

Partial fraction expansions (*Figure 3.20*) provide an approach to integrating a general rational function. Any rational function of a real variable can be written as the sum of a polynomial and a finite number of rational fractions whose denominator is the power of an **irreducible** polynomial and whose

$$\frac{f(x)}{g(x)}$$

Figure 3.20 Partial Fraction Expansion
The formula for partial fractions is given by f(x)/g(x)

numerator has a degree lower than the degree of this irreducible polynomial. Here are some common examples.

A 1st-Degree Polynomial in the Denominator

The substitution u = ax + b, du = a dx reduces the integral $\int \frac{1}{ax+b}\,dx$ to $\int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C.$

A Repeated 1st-Degree Polynomial in the Denominator

The same substitution reduces such integrals as $\int \frac{1}{(ax+b)^8}\,dx$ to

$$\int \frac{1}{u^8}\frac{du}{a} = \frac{1}{a}\int u^{-8}\,du = \frac{1}{a}\cdot\frac{u^{-7}}{(-7)} + C = \frac{-1}{7au^7} + C = \frac{-1}{7a(ax+b)^7} + C.$$

An Irreducible 2nd-Degree Polynomial in the Denominator

Next we consider integrals such as $\int \frac{x+6}{x^2-8x+25}\,dx$. The quickest way to see that the denominator $x^2 - 8x + 25$ is irreducible is to observe that its discriminant is negative. Alternatively, we can complete the square: $x^2 - 8x + 25 = (x^2 - 8x + 16) + 9 = (x-4)^2 + 9$, and observe that this sum of two squares can never be 0 while x is a real number. In order to make use of the substitution

$u = x^2 - 8x + 25$

$du = (2x - 8)\,dx$

$du/2 = (x - 4)\,dx$

we would need to find x – 4 in the numerator. So we decompose the numerator x + 6 as (x – 4) + 10, and we write the integral as $\int \frac{x-4}{x^2-8x+25}\,dx + \int \frac{10}{x^2-8x+25}\,dx$. The substitution handles the first summand, thus:

$$\int \frac{x-4}{x^2-8x+25}\,dx = \int \frac{du/2}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2-8x+25) + C.$$

Note that the reason we can discard the absolute value sign is that, as we observed earlier, $(x - 4)^2 + 9$ can never be negative.

Next we must treat the integral $\int \frac{10}{x^2-8x+25}\,dx$. With a little more algebra,

$$\int \frac{10}{x^2-8x+25}\,dx = \int \frac{10}{(x-4)^2+9}\,dx$$

$$[9pt] = \int \frac{10/9}{\left(\frac{x-4}{3}\right)^2+1}\,dx = \frac{10}{3}\arctan\left(\frac{x-4}{3}\right) + C.$$

Putting it all together,

$$\int \frac{x+6}{x^2-8x+25}\,dx = \frac{1}{2}\ln(x^2-8x+25) + \frac{10}{3}\arctan\left(\frac{x-4}{3}\right) + C.$$

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Integration Using Tables and Computers

Tables of known integrals or computer programs are commonly used for integration.

KEY POINTS

- While differentiation has easy rules by which the derivative of a complicated function can be found by differentiating its simpler component functions, integration does not.
- In books with integral tables, a compilation of a list of integrals and techniques of integral calculus can be found.
- There are several commercial softwares, such as Mathematica or Matlab, that can perform symbolic integration.

Integration is the basic operation in integral calculus. While differentiation has easy rules by which the derivative of a complicated function can be found by differentiating its simpler component functions, integration does not, so tables of known integrals are often useful. We also may have to resort to computers to perform an integral.

Integration Using Tables

A compilation of a list of integrals and techniques of integral calculus was published by the German mathematician Meyer Hirsch as early as in 1810. More extensive tables were compiled in 1858 by the Dutch mathematician David de Bierens de Haan. A new edition was published in 1862. These tables, which contain mainly integrals of elementary functions, remained in use until the middle of the 20th century. They were then replaced by the much more extensive tables of Gradshteyn and Ryzhik. Here are a few examples of integrals in these tables for logarithmic functions:

$$\int \ln ax \; dx = x \ln ax - x$$

$$\int \ln(ax + b) \; dx = \frac{(ax + b)\ln(ax + b) - ax}{a}$$

$$\int (\ln x)^2 \; dx = x(\ln x)^2 - 2x \ln x + 2x$$

$$\int (\ln x)^n \; dx = x \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} (\ln x)^k$$

$$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \sum_{k=2}^{\infty} \frac{(\ln x)^k}{k \cdot k!}$$

$$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1} \int \frac{dx}{(\ln x)^{n-1}} \qquad (\text{for } n \neq 1)\,.$$

You can certainly see that these integrals are hard to do simply "by hand."

Integration Using Computers

Computers may be used for integration in two primary ways. First, numerical methods using computers can be helpful in evaluating a definite integral. There are many methods and algorithms (*Figure 3.21*). We will briefly learn about numerical integration in another atom. Second, there are several commercial softwares, such as Mathematica or Matlab, that can perform symbolic integration.

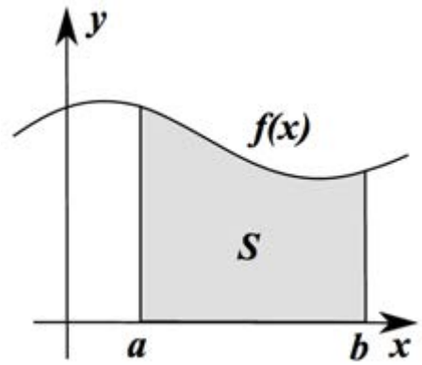

Figure 3.21 Integration

Numerical integration consists of finding numerical approximations for the value S.

EXAMPLE

Example: Mathematica's symbolic integration produces the following result:

$$\int log(1-x^2)dx = -2x - log(x-1) + log(1+x) + xlog(1-x^2)] .$$

These programs know how to perform almost any integral that can be done analytically or in terms of standard mathematical functions.

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Approximate Integration

The trapezoidal rule (also known as the trapezoid rule or trapezium rule) is a technique for approximating the definite integral.

KEY POINTS

- The trapezoidal rule works by approximating the region under the graph of the function f(x) as a trapezoid and calculating its area: $\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}$.
- For a domain discretized into N equally spaced panels, or N +1 grid points (1, 2,..., N+1), where the grid spacing is h=(b-a)/N, the approximation to the integral becomes $\int_a^b f(x)\,dx = \frac{b-a}{2N}(f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_N) + f(x_{N+1}))$
- In two and higher dimensions, where simple approximation methods become prohibitively expensive in terms of computational effort, one may use other methods such as the Monte Carlo method.

Numerical integration, in some instances also known as numerical quadrature, asks for the value of a definite integral. Popular methods use one of the Newton–Cotes formulas (such as midpoint rule or Simpson's rule) or Gaussian quadrature. These methods rely on a "divide and conquer" strategy, whereby an integral on a

relatively large set is broken down into integrals on smaller sets. In higher dimensions, where these methods become prohibitively expensive in terms of computational effort, one may use other methods such as the Monte Carlo method. Here, we will study a very simple approximation technique, called a **trapezoidal** rule.

Trapezoidal rule

The trapezoidal rule (also known as the trapezoid rule or trapezium rule) is a technique for approximating the definite integral $\int_a^b f(x)\,dx$. The trapezoidal rule works by approximating the region under the graph of the function f(x) as a trapezoid and calculating its area. It follows that $\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}$ (*Figure 3.22*).

The trapezoidal rule tends to become extremely accurate when periodic functions are integrated over their periods.

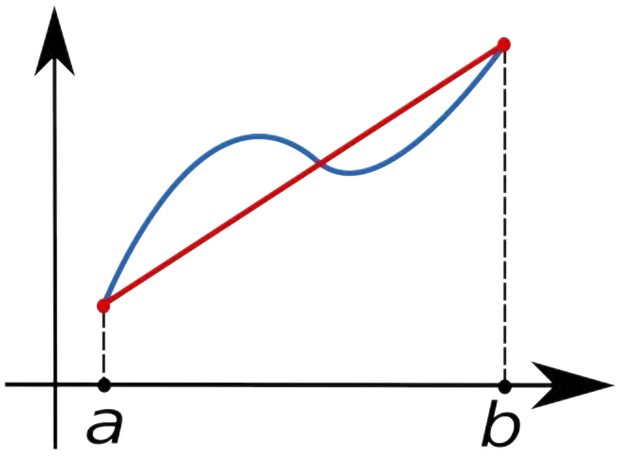

Figure 3.22
Approximation by Linear Functions
The function f(x) (in blue) is approximated by a linear function (in red).

Numerical Implementation of the Trapezoidal Rule

For a domain discretized into N equally spaced panels, or N+1 grid points (1, 2,..., N+1), where the grid spacing is h=(b-a)/N, the approximation to the integral becomes

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{k=1}^{N}\left(f(x_{k+1})+f(x_k)\right)$$

$$=\frac{b-a}{2N}(f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_N)+f(x_{N+1})).$$

Although the method can adopt a nonuniform grid as well, this example used a uniform grid for the the approximation.

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Improper Integrals

An Improper integral is the limit of a definite integral as an endpoint of the integral interval approaches either a real number or ∞ or −∞.

KEY POINTS

- An improper integral may be a limit of the form $\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$, $\lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x$.
- It could also be a limit of the form $\lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}y$, $\lim_{c\to a^+}\int_c^b f(x)\,\mathrm{d}x$, in which one takes a limit in one or the other (or sometimes both) endpoints.
- It is often necessary to use improper integrals in order to compute a value for integrals which may not exist in the conventional sense (as a Riemann integral, for instance) because of a singularity in the function, or an infinite endpoint of the domain of integration.

An improper integral is the limit of a **definite integral** as an endpoint of the interval(s) of integration approaches either a specified real number or ∞ or −∞ or, in some cases, as both endpoints approach limits. Such an integral is often written symbolically just like a standard definite integral, perhaps with infinity as a limit of integration. But that conceals the limiting process.

Specifically, an improper integral is a limit of the form $\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$, $\lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x$ (*Figure 3.23*), or of the form $\lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}y$, $\lim_{c\to a^+}\int_c^b f(x)\,\mathrm{d}x$ (*Figure 3.24*), in which one takes a limit in one or the other (or sometimes both) endpoints. Integrals are also improper if the integrand is undefined at an interior point of the domain of integration, or at multiple such points. It is often necessary to use improper integrals in order to compute a value for integrals which may not exist in the conventional sense (as a Riemann integral, for instance) because of a singularity in the function, or an infinite endpoint of the domain of integration.

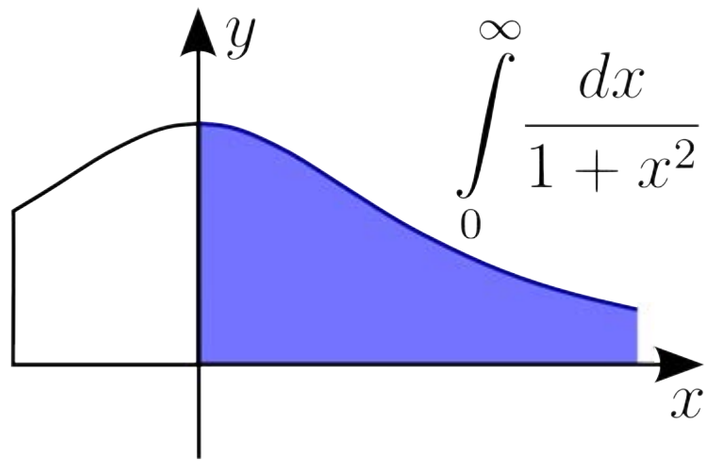

Figure 3.23
Improper Integral of the First Kind
The integral may need to be defined on an unbounded domain.

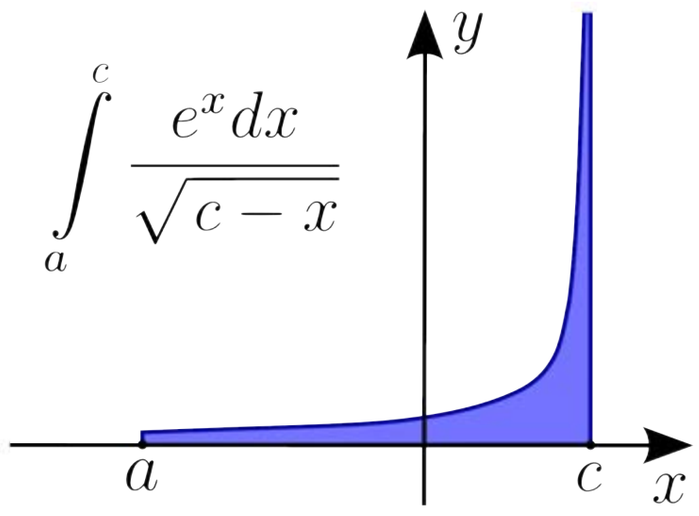

Figure 3.24
Improper Integral of the Second Kind
An improper Riemann integral of the second kind. The integral may fail to exist because of a vertical asymptote in the function.

definition requires that both the domain of integration and the integrand be bounded). However, the improper integral does exist if understood as the limit

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0^+} (2\sqrt{1} - 2\sqrt{a}) = 2.$$

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Examples

1. The original definition of the Riemann integral does not apply to a function such as $1/x^2$ on the interval $[1, \infty)$, because in this case the domain of integration is unbounded. However, the Riemann integral can often be extended by continuity, by defining the improper integral instead as a limit

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1}\right) = 1.$$

2. The narrow definition of the Riemann integral also does not cover the function $1/\sqrt{x}$ on the interval $[0, 1]$. The problem here is that the **integrand** is unbounded in the domain of integration (the

Numerical Integration

Numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral.

KEY POINTS

- The basic problem considered by numerical integration is to compute an approximate solution to a definite integral: $\int_a^b f(x)\,dx$.
- There are several reasons for carrying out numerical integration. It could be due to the specific nature of the function (to be integrated) or its antiderivatives.
- A large class of quadrature rules can be derived by constructing interpolating functions which are easy to integrate. Typically these interpolating functions are polynomials. Midpoint method and trapezoidal method are simple examples.

Numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral, and, by extension, the term is also sometimes used to describe the numerical solution of differential equations. This article focuses on calculation of definite integrals. The term numerical quadrature (often abbreviated to quadrature) is more or less a synonym for numerical integration, especially as applied to one-dimensional integrals. Numerical integration over more than one dimension is sometimes described as cubature, although the meaning of quadrature is understood for higher dimensional integration as well.

The basic problem considered by numerical integration is to compute an approximate solution to a definite integral: $\int_a^b f(x)\,dx$. If f(x) is a smooth well-behaved function, integrated over a small number of dimensions and the limits of integration are bounded, there are many methods of approximating the integral with arbitrary precision.

Reasons for numerical integration

1. There are several reasons for carrying out numerical integration. The integrand f(x) may be known only at certain points, such as when obtained by sampling. Some embedded systems and other computer applications may need numerical integration for this reason.

2. A formula for the integrand may be known, but it may be difficult or impossible to find an antiderivative which is an elementary function. An example of such an integrand is f(x)

= $\exp(-x^2)$, the antiderivative of which (the error function, times a constant) cannot be written in elementary form.

3. It may be possible to find an antiderivative symbolically, but it may be easier to compute a numerical approximation than to compute the antiderivative. That may be the case if the antiderivative is given as an infinite series or product, or if its evaluation requires a special function which is not available.

Methods for One-Dimensional Integrals

A large class of quadrature rules can be derived by constructing interpolating functions which are easy to integrate. Typically these interpolating functions are polynomials. The simplest method of this type is to let the interpolating function be a constant function (a polynomial of degree zero) which passes through the point ((a+b)/2, f((a+b)/2)). This is called the midpoint rule or rectangle rule:

$\int_a^b f(x)\,dx \approx (b-a) f\left(\frac{a+b}{2}\right)$ (*Figure 3.25*). The interpolating function may be an affine function (a polynomial of degree 1) which passes through the points (a, f(a)) and (b, f(b)). This is called the trapezoidal rule (*Figure 3.26*).

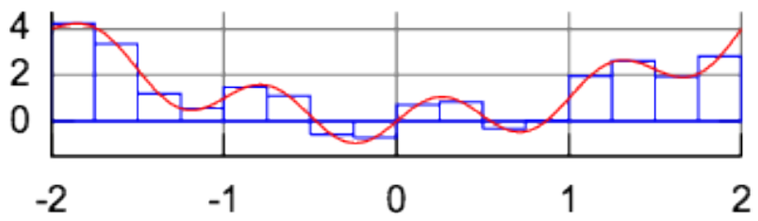

Figure 3.25
Rectangle Rule
Illustration of the rectangle rule.

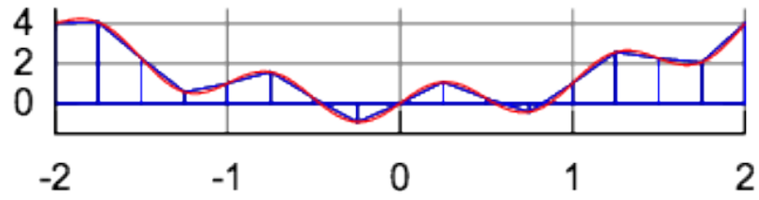

Figure 3.26
Trapezoidal Rule
Illustration of the trapezoidal rule.

For either one of these rules, we can make a more accurate approximation by breaking up the interval [a, b] into some number n of subintervals, computing an approximation for each subinterval, then adding up all the results. This is called a composite rule, extended rule, or iterated rule. For example, the composite trapezoidal rule can be stated as

$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2} + \sum_{k=1}^{n-1}\left(f\left(a+k\frac{b-a}{n}\right)\right) + \frac{f(b)}{2}\right)$, where the subintervals have the form [k h, (k+1) h], with h = (b−a)/n and k = 0, 1, 2,..., n−1.

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Section 3

Further Applications of Integration

Arc Length and Surface Area

Area of a Surface of Revolution

Physics and Engineering: Fluid Pressure and Force

Physics and Engineering: Center of Mass

Applications to Economics and Biology

Probability

Taylor Polynomials

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Arc Length and Surface Area

Infinitesimal calculus provides us general formulas for the arc length of a curve and the surface area of a solid.

KEY POINTS

- For a curve represented by f(x) in range [a,b], arc length s is give as $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.
- If a curve is defined parametrically by x = X(t) and y = Y(t), then its arc length between t = a and t = b is $s = \int_a^b \sqrt{[X'(t)]^2 + [Y'(t)]^2}\, dt$.
- For rotations around the x- and y-axes, surface areas A_x and A_y are given, respectively, as the following:

$$A_x = \int 2\pi y\, ds,\ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$A_y = \int 2\pi x\, ds,\ ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

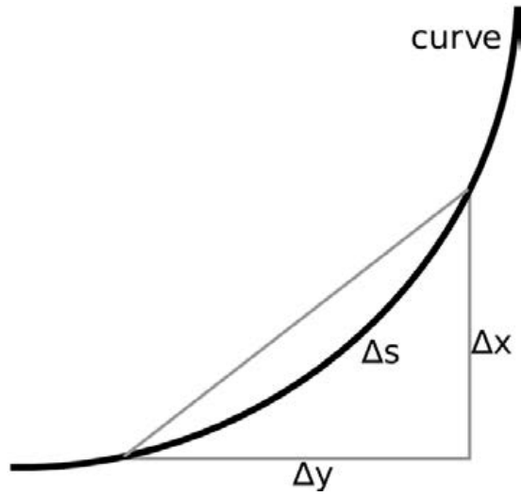
Determining the length of an irregular arc segment is also called rectification of a **curve**. Historically, many methods have been used for specific curves. The advent of infinitesimal calculus led to a general formula, which we will learn in this atom. We will also use integration to calculate the **surface area** of a three-dimensional object.

Arc Length

Consider a real function f(x) such that f(x) and $f'(x) = \frac{dy}{dx}$ (its derivative with respect to x) are continuous on [a, b]. The length s of the part of the graph of f between x = a and x = b can be found as follows:

Consider an infinitesimal part of the curve ds (or consider this as a limit in which the change in s approaches ds). According to Pythagoras's theorem (*Figure 3.27*) $ds^2 = dx^2 + dy^2$, from which $\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}$

Figure 3.27 Approximating Deltas



For a small piece of curve, Δs can be approximated with the Pythagorean theorem.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

If a curve is defined parametrically by x = X(t) and y = Y(t), then its arc length between t = a and t = b is $s = \int_a^b \sqrt{[X'(t)]^2 + [Y'(t)]^2}\, dt$. This is more clearly a consequence of the distance formula, where instead of a Δx and Δy, we take the limit. A useful mnemonic is

$$s = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Surface Area

For rotations around the x- and y-axes, surface areas A_x and A_y are given, respectively, as the following:

$$A_x = \int 2\pi y\, ds,\ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$A_y = \int 2\pi x\, ds,\ ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example: For a circle $f(x) = \sqrt{1 - x^2}, 0 \le x \le 1$

1. Calculate the arc length.

The curve can be represented parametrically as $x = sin(t), y = cos(t)$ for $0 \le t \le \frac{\pi}{2}$. Therefore, $s = \int_0^{\frac{\pi}{2}} \sqrt{cos^2(t) + sin^2(t)} = \frac{\pi}{2}$

2. Calculate the surface area of the solid obtained by rotating f(x) around that x-axis.

$$A_x = \int_0^1 2\pi\sqrt{1 - x^2} \cdot \sqrt{1 + (\frac{-x}{\sqrt{1 - x^2}})^2}\, dx = 2\pi$$

EXAMPLE

For a circle:

$$A_x = \int_0^1 2\pi\sqrt{1 - x^2} \cdot \sqrt{1 + (\frac{-x}{\sqrt{1 - x^2}})^2}\, dx = 2\pi$$

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Area of a Surface of Revolution

If the curve is described by the function y = f(x) (a≤x≤b), the area A_y is given by the integral

$A_x = 2\pi \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx$ for revolution around the x-axis.

KEY POINTS

- A surface of revolution is a surface in Euclidean space created by rotating a curve around a straight line in its plane, known as the axis.
- If the curve is described by the parametric functions x(t), y(t), with t ranging over some interval [a,b] and the axis of revolution is the y-axis, then the area Ay is given by the integral $A_y = 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.
- If the curve is described by the function y = f(x), a ≤ x ≤ b, then the integral becomes $A_x = 2\pi \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx$ for revolution around the x-axis.
- Examples of surfaces generated by a straight line are cylindrical and conical surfaces when the line is co-planar with the axis.

Figure 3.28 Surface of Revolution

A portion of the curve x=2+cos z rotated around the z axis (vertical in the figure).

A surface of **revolution** is a surface in **Euclidean space** created by rotating a curve around a straight line in its plane, known as the axis (*Figure 3.28*). Examples of surfaces generated by a straight line are cylindrical and conical surfaces when the line is co-planar with the axis, as well as hyperboloids of one sheet when the line is skew to the axis. A circle that is rotated about a diameter generates a sphere, and if the circle is rotated about a co-planar axis other than the diameter it generates a **torus**.

If the curve is described by the parametric functions x(t), y(t), with t ranging over some interval [a,b] and the axis of revolution is the y-axis, then the area A_y is given by the integral

$A_y = 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, provided that x(t) is never

negative between the endpoints a and b. The quantity

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ comes from the Pythagorean theorem and represents a small segment of the arc of the curve, as in the arc length formula. Likewise, when the axis of rotation is the x-axis, and provided that y(t) is never negative, the area is given by

$$A_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \text{ (1)}$$

If the curve is described by the function y = f(x), a ≤ x ≤ b, then the integral becomes

$A_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2}\, dx$ for revolution around the x-axis, and $A_y = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ for revolution around the y-axis (a ≤ y ≤ b). These come from the above formula (1).

Example

The spherical surface with a radius r is generated by the curve x(t) =rsin(t), y(t) = rcos(t), when t ranges over [0,π]. Its area is therefore

$$A = 2\pi \int_0^{\pi} r \sin(t) \sqrt{\left(r\cos(t)\right)^2 + \left(r\sin(t)\right)^2}\, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin(t)\, dt$$

$$= 4\pi r^2.$$

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Physics and Engineering: Fluid Pressure and Force

Pressure is give as the following: $p = \frac{F}{A}$ or $p = \frac{dF_n}{dA}$, where p is the pressure, F is the normal force, and A is the area of the surface on contact.

KEY POINTS

- The pressure is the scalar proportionality constant that relates the two normal vectors $d\mathbf{F}_n = -p\,d\mathbf{A} = -p\,\mathbf{n}\,dA$.
- For fluids near the surface of the earth, the formula may be written as ρ is the density of the fluid, g is the gravitational acceleration, and h is the depth of the liquid in meters.
- Total force that the fluid pressure gives rise to is calculated as $\mathbf{F_n} = -\left(\int \rho g h\, dA\right)\mathbf{n}$.

Pressure (p) is force per unit area applied in a direction perpendicular to the surface of an object (*Figure 3.29*). While pressure may be measured in any unit of force divided by any unit of area, the SI unit of pressure (the newton per square metre) is called the pascal (Pa).

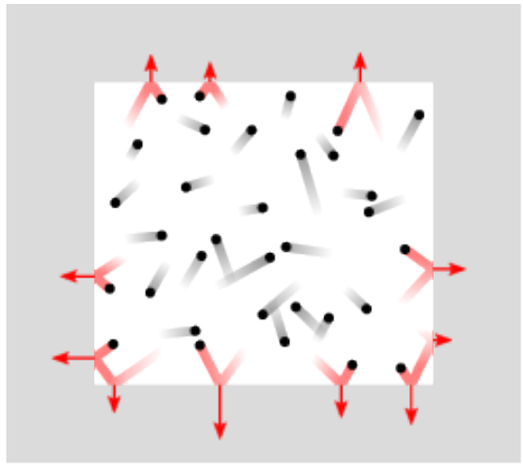

Figure 3.29 Fluid Pressure and Force Pressure as exerted by particle collisions inside a closed container.

Mathematically, $p = \frac{F}{A}$, where p is the pressure, F is the normal force, and A is the area of the surface on contact. Pressure is a scalar quantity. It relates the vector surface element (a vector normal to the surface) with the normal force acting on it. The pressure is the scalar proportionality constant that relates the two normal vectors: $d\mathbf{F}_n = -p\,d\mathbf{A} = -p\,\mathbf{n}\,dA$. The subtraction (-) sign comes from the fact that the force is considered towards the surface element while the normal vector points outward. The total force normal to the contact surface would be $\mathbf{F_n} = \int d\mathbf{F_n} = -\int p\,d\mathbf{A} = -\int p\,\mathbf{n}\,dA$ (1).

Pressure is an important quantity in the studies of fluid (for example, in weather forecast). For fluids near the surface of the earth, the formula may be written as $p = \rho g h$, where p is the pressure, ρ is the density of the fluid, g is the gravitational acceleration, and h is the depth of the liquid in meters. Using this expression, we can calculate the total force that the fluid pressure gives rise to $\mathbf{F_n} = -\left(\int \rho g h\, dA\right)\mathbf{n}$. This equation, for example, can be

used to calculate the total force on a submarine submerged in the sea.

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Physics and Engineering: Center of Mass

For a continuous mass distribution, the position of center of mass is given as $\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r})\mathbf{r}dV$.

KEY POINTS

- In physics, the center of mass (COM) of a distribution of mass in space is the unique point at which the weighted relative position of the distributed mass sums to zero.
- In the case of a system of particles P_i, i = 1, ..., n , each with mass, m_i, which are located in space with coordinates r_i, i = 1, ..., n, the coordinates R of the center of mass is $\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$.
- If the mass distribution is continuous with respect to the density, ρ(r), within a volume, V, then it follows that $\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r})\mathbf{r}dV$.

Center of Mass

In physics, the center of mass (COM) of a mass or object in space is the unique point at which the weighted relative position of the

distributed mass sums to zero. In this case, the distribution of mass is balanced around the center of mass and the average of the weighted position coordinates of the distributed mass defines its coordinates. Calculations in mechanics are simplified when formulated with respect to the COM. Referring to *Figure 3.30*, we see that the two objects are rotating around their center of mass.

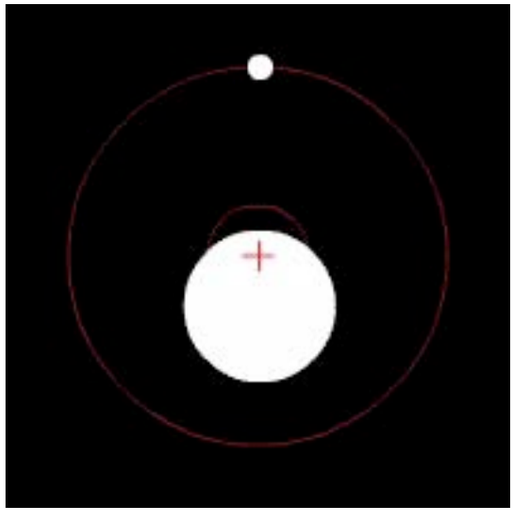

Figure 3.30 Two Bodies and the COM

Two bodies orbiting the COM located inside one body. COM can be defined for both discrete and continuous systems.

System of Particles

In the case of a system of particles P_i, i = 1, ..., n , each with a mass, m_i, which are located in space with coordinates r_i, i = 1, ..., n , the coordinates R of the center of mass satisfy the following condition: $\sum_{i=1}^{n} m_i(\mathbf{r}_i - \mathbf{R}) = 0$. Solve this equation for R to obtain the formula $\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$, where M is the sum of the masses of all of the particles.

Continuous Distribution

If the mass distribution is continuous with respect to the density, ρ(r), within a volume, V, then the integral of the weighted position coordinates of the points in this volume relative to the center of mass, R, is zero, that is: $\int_V \rho(\mathbf{r})(\mathbf{r} - \mathbf{R})dV = 0$. Solve this equation for the coordinates R to obtain: $\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r})\mathbf{r}dV$ where M is the total mass in the volume. If a continuous mass distribution has uniform density, which means ρ is constant, then the center of mass is the same as the centroid of the volume. Note that you can derive the same result by making the following substitution in the definition of the COM in discrete systems: $\sum \to \int, m_i \to dm = \rho(r)dV, \mathbf{r_i} \to \mathbf{r}$.

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Applications to Economics and Biology

Calculus has broad applications in diverse fields of science; examples of integration can be found in economics and biology.

KEY POINTS

- Consumer surplus is thus the definite integral of the demand function with respect to price, from the market price to the maximum reservation price $CS = \int_{P_{mkt}}^{P_{max}} D(P)\,dP$.
- The total flux of blood though a vessel with a radius R can be expressed as $F = \int_0^R 2\pi r\, v(r)\,dr$, where v(r) is the velocity of blood at r.
- Calculus, in general, has broad applications in diverse fields of science.

Calculus, in general, has a broad applications in diverse fields of science, finance, and business. In this atom, we will see some examples of applications of integration in economics and biology.

Consumer Surplus

In mainstream economics, economic **surplus** (also known as total welfare or Marshallian surplus) refers to two related quantities. Consumer surplus is the monetary gain obtained by consumers; they are able to buy something for less than they had planned on spending. Producer surplus is the amount that producers benefit from selling at a market price that is higher than their lowest price, thereby making more profit (*Figure 3.31*).

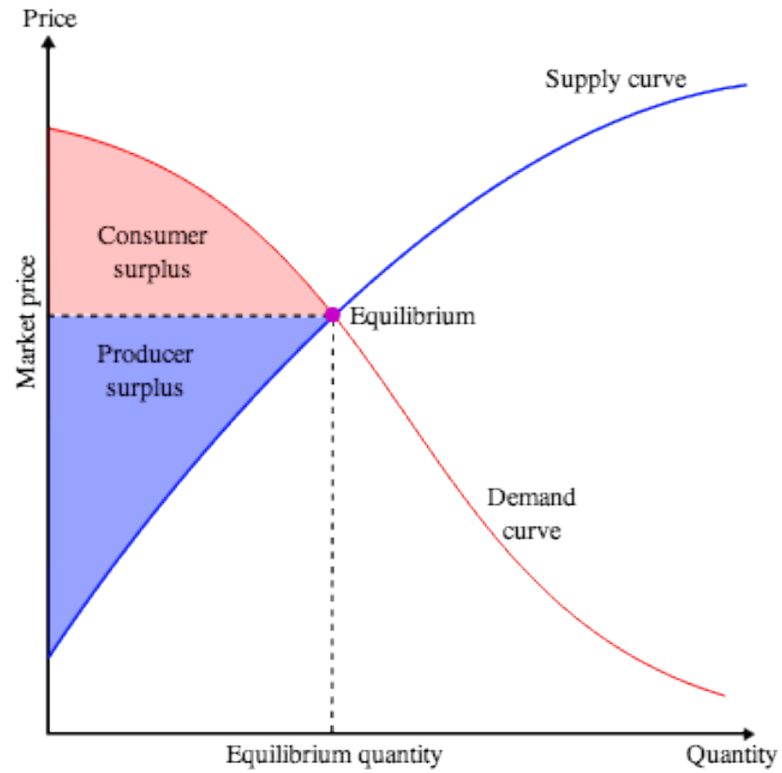

Figure 3.31 Supply and Demand Chart
Graph illustrating consumer (red) and producer (blue) surpluses on a supply and demand chart.

In calculus terms, consumer surplus is the derivative of the definite integral of the demand function with respect to price, from the

market price to the maximum reservation price (i.e. the price-intercept of the demand function): $CS = \int_{P_{mkt}}^{P_{max}} D(P)dP$ where $D(P_{max}) = 0$ (D(P) is a demand curve as a function of price).

Blood Flow

The human body is made up of several processes, all carrying out various functions, one of which is the continuous running of blood in the cardiovascular system. If we wanted, we could obtain a general expression for the volume of blood across a cross section per unit time (a quantity called flux). Since we can assume that there is a cylindrical symmetry in the blood vessel, we first consider the volume of blood passing through a ring with inner radius r and outer radius r+dr per unit time (dF). $dF = (2\pi r\, dr)\, v(r)$, where v(r) is the speed of blood at radius r. Here, $2\pi r\, dr$ is the area of the ring. Therefore, the total flux F is written as $F = \int_0^R 2\pi r\, v(r)\, dr$, where R is the radius of the blood vessel. Once we have an (approximate) expression for v(r) (*Figure 3.32*), we can calculate the **flux** from the integral.

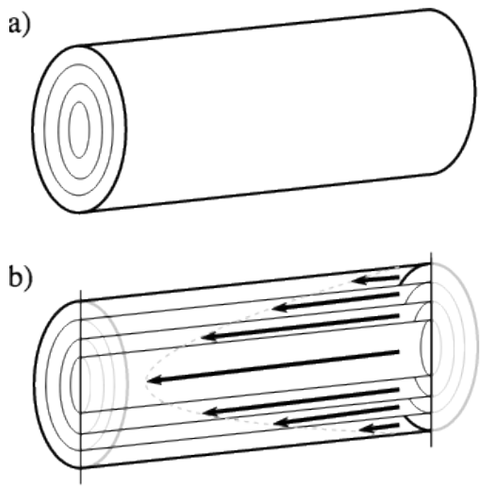

Figure 3.32
Blood Flow
(a) A tube; (b) The blood flow close to the edge of the tube is slower than that near the center.

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Probability

Probability density function describes the relative likelihood, or probability, that a given variable will take on a value.

KEY POINTS

- The probability of X to be in a range [a,b] is given as $P[a \leq X \leq b] = \int_a^b f(x)\,\mathrm{d}x$, where f(x) is the probability density function in this case.
- The integral of the partial distribution function over the entire range of the variable is 1.
- The standard normal distribution has probability density $f(X;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$.

Integration is commonly used in statistical analysis, especially when a random variable takes a continuum value. Here, we will learn what probability distribution function is and how it functions with regard to integration.

In probability theory, a **probability density function** (pdf), or density of a continuous random variable, is a function that describes the relative likelihood for this random variable to take on a given value. The probability for the random variable to fall within a particular region is given by the integral of this variable's probability density over the region. The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one.

Probability Density Function

A probability density function is most commonly associated with absolutely continuous univariate distributions.

- For a continuous random variable X, the probability of X to be in a range [a,b] is given as $P[a \leq X \leq b] = \int_a^b f(x)\,\mathrm{d}x$, where f(x) is the probability density function in this case.
- The integral of the pdf in the range $[-\infty, \infty]$ is 1: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$
- The expected value of X (if it exists) can be calculated as $E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$

Example: Normal Distribution

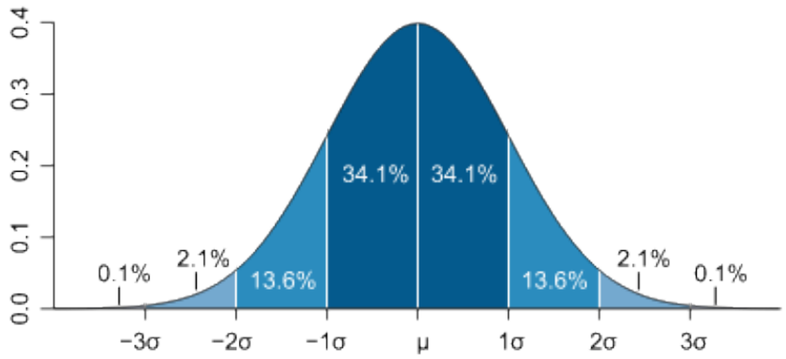

Figure 3.33
Probability Distribution Function
Probability distribution function of a normal (or Gaussian) distribution, where mean $\mu = 0$ and variance $\sigma^2 = 1$.

The standard normal distribution has probability density $f(X;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$. This probability distribution has the mean and variance, denoted by μ and σ^2, respectively. As shown in *Figure 3.33*, the probability to have x in the range $[\mu - \sigma, \mu + \sigma]$ can be calculated from the integral $\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \approx 0.682.$

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Taylor Polynomials

A Taylor series is a representation of a function as an infinite sum of terms calculated from the values of the function's derivatives.

KEY POINTS

- The Taylor series of a real or complex-valued function f(x) that is infinitely differentiable in a neighborhood of a real or complex number a is the power series
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n .$
- Any finite number of initial terms of the Taylor series of a function is called a Taylor **polynomial**.
- Taylor series can be used to evaluate an integral when there is no other integration technique available (other than numerical integration).

Taylor **series** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The Taylor series of a real or complex-valued function f(x) that is infinitely differentiable in a neighborhood of a real or complex number a is the power series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, where n! denotes the factorial of n and $f^{(n)}a$

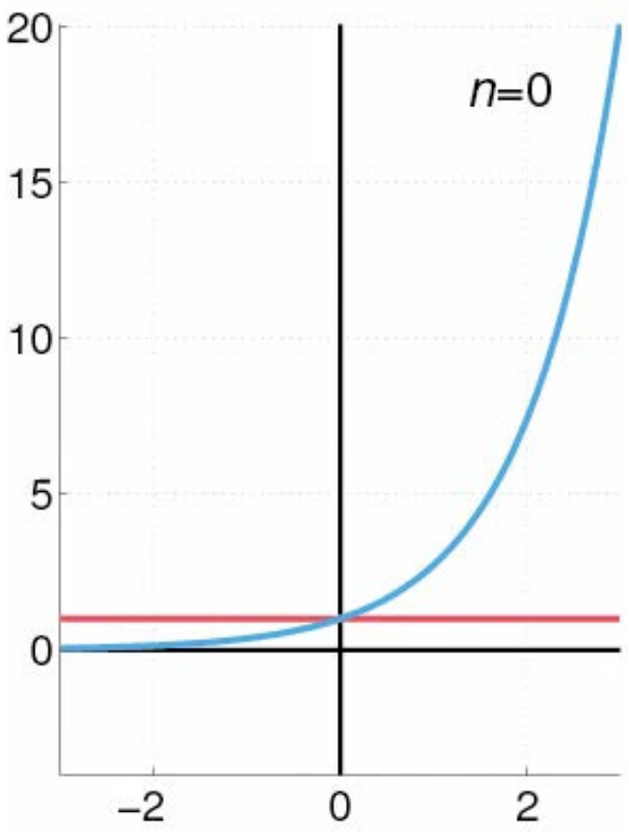

Figure 3.34 Exponential Function as a Taylor Series

The exponential function (in blue) and the sum of the first 9 terms of its Taylor series at 0 (in red).

denotes the n-th derivative of f evaluated at the point x=a. Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial (*Figure 3.34*).

Example: The Taylor series for the exponential function e^x at a = 0 is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$.

Using Taylor Series to Evaluate an Integral

Taylor series can be used to evaluate an integral when there is no other integration technique available (of course, other than numerical integration). Let's assume that the integration of a function (f(x)) cannot be performed analytically. To evaluate the integral $I = \int_a^b f(x)\,dx$, we can Taylor-expand f(x) and perform integration on individual terms of the series. Since $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we get

$I = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_a^b x^n\,dx = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{(n+1)!} (b^{n+1} - a^{n+1})$. Therefore, as long as Taylor expansion is possible and the infinite sum converges, the definite integral (I) can be evaluated.

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Chapter 4

Differential Equations, Parametric Equations, and Sequences and Series

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Section 1

Differential Equations

Solving Differential Equations

Models Using Differential Equations

Direction Fields and Euler's Method

Separable Equations

Logistic Equations and Population Grown

Linear Equations

Predator-Prey Systems

https://www.boundless.com/calculus/differential-equations-parametric-equations-and-sequences-and-series/differential-equations/

Solving Differential Equations

Differential equations are solved by finding the function for which the equation holds true.

KEY POINTS

- The order of a differential equation is determined by the highest-order derivative; the degree is determined by the highest power on a variable.
- The higher the order of the differential equation, the more arbitrary constants need to be added to the general solution. A first-order equation will have one, a second-order two, and so on.
- A particular solution can be found by assigning values to the arbitrary constants to match any given constraints.

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the **function** itself to its **derivatives** of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines.

Differential equations take a form similar to $f(x) + f'(x) = 0$,

where f' is f-prime, the derivative of f. As you can see, such an equation relates a function f(x) to its derivative. Solving the differential equation means solving for the function f(x).

Figure 4.1
Differential Equation

$$d^2y/dx^2 + xy = x^3y^3$$

An example of a differential equation

The order of a differential equation is determined by the highest order derivative; the degree is determined by the highest power on a variable. For example, the differential equation shown in *Figure 4.1* is of second-order, third-degree, and the one above is of first-order, first-degree.

Solving the above equation shows that f(x) is equal to the negative of its derivative; therefore, the function f(x) must be e^{-x}, as the derivative of this function equals the negative of the original function. The derivative of e^{-x} equals e^{-x}, so this must be the answer.

A complete solution contains a number of arbitrary constants equal to the order of the equation. (To solve an nth order differential equation, you have to perform n integrations, and each time you integrate, you have to introduce an arbitrary constant.) Since our example above is a first-order equation, it will have one arbitrary constant in the complete solution. Therefore, the general solution is $f(x) = Ce^{-x}$ where C stands for an arbitrary constant. You can see that the differential equation still holds true with this constant. A particular solution is obtained by assigning specific values to the constants in the general solution.

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Models Using Differential Equations

Differential equations can be used to model a variety of physical systems.

KEY POINTS

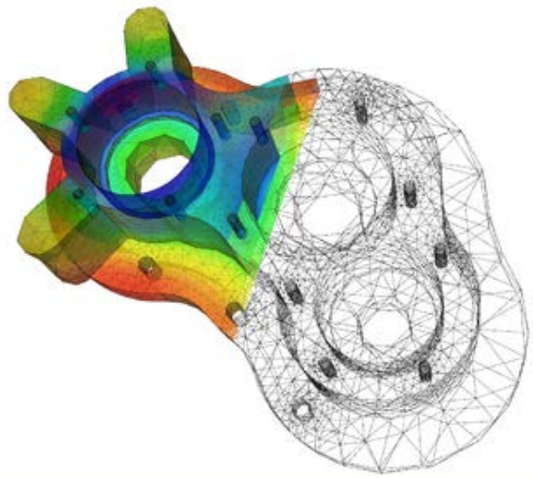
- Many systems can be well understood through differential equations.
- Mathematical models of differential equations can be used to solve problems and generate models.
- An example of such a model is the differential equation governing radioactive decay.

Differential equations are very important in the mathematical modeling of physical systems.

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give

rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena. As an example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond. All of them may be described by the same second-order partial-differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat is governed by another second-order partial differential equation, the heat equation (*Figure 4.2*).

Figure 4.2 Visual Model of Heat Transfer



Visualization of heat transfer in a pump casing, created by solving the heat equation. Heat is being generated internally in the casing and being cooled at the boundary, providing a steady state temperature distribution.

A good example of a physical system modeled with differential equations is radioactive **decay** in physics.

Over time, radioactive elements decay. The half-life—t1/2, is the time taken for the activity of a given amount of a radioactive substance to decay to half of its initial value. The mean lifetime— τ, "tau" is the average lifetime of a radioactive particle before decay. The decay constant— λ, "lambda" is the inverse of the mean lifetime.

We can combine these quantities in a differential equation to determine the activity of the substance. For a number of radioactive particles N, the activity A, or number of decays per time is given by:

$$A = -\frac{dN}{dt} = \lambda N$$

a first order differential equation.

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Direction Fields and Euler's Method

Direction fields and Euler's method are ways of visualizing and approximating the solutions to differential equations.

KEY POINTS

- Direction fields, or slope fields, are graphs where each point x,y has a slope.
- Euler's method is a way of approximating solutions to differential equations by assuming that the slope at a point is the same as the slope between that point and the next point.
- Euler's method gives approximate solutions to differential equations, and the smaller the distance between the chosen points, the more accurate the result.

Direction Fields

Direction fields, also known as slope fields, are graphical representations of the solution to a first order differential equation. They can be achieved without solving the differential equation analytically, and serve as a useful way to visualize the solutions.

The slope field is traditionally defined for the following type of differential equations: $y' = f(x)$ It can be viewed as a creative way to plot a real-valued function of two real variables as a planar picture. See *Figure 4.3* for an example.

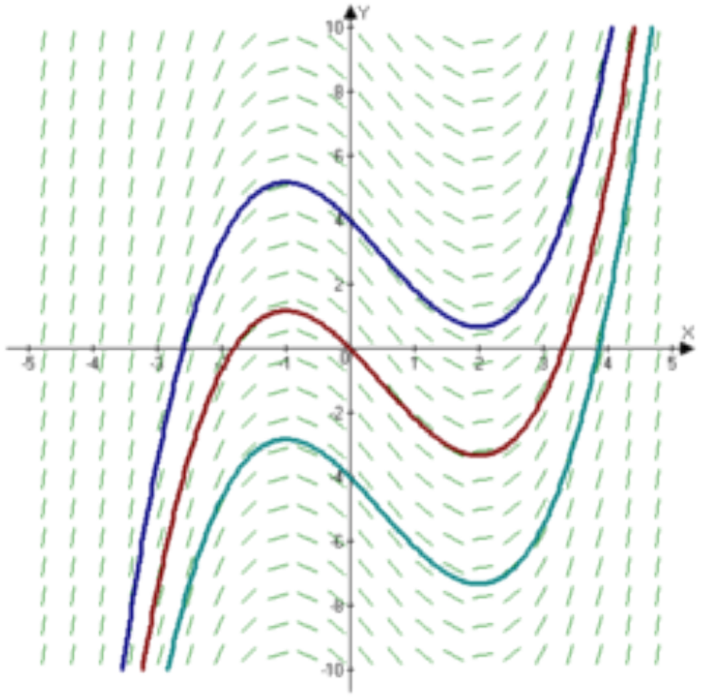

Figure 4.3 Example slope field

The slope field of $dy/dx = x^2 - x - 2$, with the blue, red, and turquoise lines being $(x^3/3) - (x^2/2) - 2x + 4$, $(x^3/3) - (x^2/2) - 2x$, and $(x^3/3) - (x^2/2) - 2x - 4$, respectively.

Specifically, for a given pair, a vector with the components is drawn at the point x,y on the x,y-plane. Sometimes, the vector is normalized to make the plot more pleasing to the human eye. A set of pairs x,y making a rectangular grid is typically used for the drawing. An isocline (a series of lines with the same slope) is often used to supplement the slope field. In an equation of the form, the isocline is a line in the x,y-plane obtained by setting f(x,y) equal to a constant.

Euler's Method

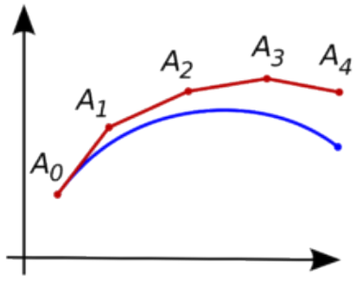

Figure 4.4 Euler's Method
Illustration of the Euler method. The unknown curve is in blue and its polygonal approximation is in red.

Consider the problem of calculating the shape of an unknown curve which starts at a given point and satisfies a given differential equation. Here, a differential equation can be thought of as a formula by which the slope of the tangent line to the curve can be computed at any point on the curve, once the position of that point has been calculated.The idea is that while the curve is initially unknown, its starting point, which we denote by A_0, is known (see *Figure 4.4*). Then, from the differential equation, the slope to the curve at A_0 can be computed, and thus, the tangent line.

Take a small step along that tangent line up to a point, A_1. Along this small step, the slope does not change too much, so A_1 will be close to the curve. If we pretend that A_1 is still on the curve, the same reasoning we used for the above point, A_0, can be applied. After several steps, a polygonal curve is computed. In general, this curve does not diverge too far from the original unknown curve, and the error between the two curves can be made small if the step size is small enough and the interval of computation is finite.

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Separable Equations

Separable differential equations are equations wherein the variables can be separated.

KEY POINTS

- Separable equations are of the form $M(y)\frac{dy}{dx} = N(x)$.
- Separable equations are among the easiest differential equations to solve.
- To solve, collect all terms that contain the same variables to one side and integrate through.

Non-linear **differential equations** come in many forms. One of these forms is separable equations. A differential equation that is separable will have several properties which can be exploited to find a solution.

A separable equation is a differential equation of the following form:

$$N(y)\frac{dy}{dx} = M(x)$$

Variables are separable if the differential equation can be expressed as f(x)dx + g(y)dy = 0, where f(x) is a function of x alone, and g(y) is a function of y alone. These are the easiest differential equations to solve. They can be integrated to give ∫f(x)dx + ∫g(y)dy = c, where c is an arbitrary constant.

To separate the equations means to move all the x's and y's to the opposite sides of the equation.

A general approach to solving separable equations is as follows:

- Clear fractions. If the equation involves derivatives, multiply through by the differential of the independent variable.
- Collect all terms containing the same differential into a single term.
- Integrate each part separately. Do not forget to add constants to equations after integrating. This ensures that the solution is of the general form.
- Simplify the expression, by combining terms, converting logarithms to exponents, and using the simplest symbol for arbitrary constants.

After simplifying you will have the general form of the equation. A particular solution to the equation will depend on the choice of the arbitrary constants you obtained when integrating.

For example, consider the time-independent Schrödinger equation (*Figure 4.5*):

$$[-\nabla^2 + V(x)]\psi(x) = E\psi(x).$$

If the function $V(x)$ in three dimensions is of the form

$$V(x_1, x_2, x_3) = V_1(x_1) + V_2(x_2) + V_3(x_3)$$

then it turns out that the problem can be separated into three one-dimensional ordinary differential equations for functions $\psi_1(x_1), \psi_2(x_2), \psi_3(x_3)$ and the final solution can be written as follows: $\psi(x) = \psi_1(x_1) * \psi_2(x_2) * \psi_3(x_3)$.

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Figure 4.5 Non-Relativistic Schrödinger Equation

A wave function which satisfies the non-relativistic Schrödinger equation with V=0. This corresponds to a particle traveling freely through empty space. The real part of the wave function is plotted here.

Logistic Equations and Population Grown

A logistic equation is a differential equation which can be used to model population growth.

KEY POINTS

- The logistic function initially grows exponentially before slowing down as it reaches a ceiling.
- This behavior makes it a good model for population growth, since populations initially grow rapidly but tend to slow down due to eventual lack of resources.
- Varying the parameters in the equation can simulate various environmental factors which impact population growth.

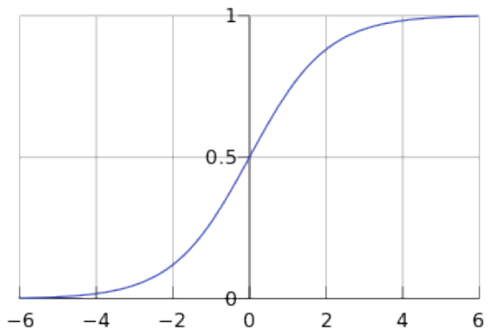
The logistic function is the solution of the simple first-order **non-linear differential equation**

$$\frac{d}{dt}P(t) = P(t)(1 - P(t))$$

with boundary condition P(0) = 1/2.

The derivative is 0 at P = 0 or 1, and the derivative is positive for P between 0 and 1 and negative for P above 1 or less than 0 (though negative populations do not generally accord with a physical

Figure 4.6 Logistic Curve



The standard logistic curve. It can be used to model population growth because of the limiting effect scarcity has on the growth rate. This is represented by the ceiling past which the function ceases to grow.

model). This yields an unstable equilibrium at 0 and a stable equilibrium at 1, and thus for any value of P greater than 0 and less than 1, P grows to 1.One may readily find the (symbolic) solution to be

$$P(t) = \frac{e^t}{e^t + e^c}$$

Choosing the constant of integration e^c = 1 gives the other well-known form of the definition of the logistic curve:

$$P(t) = \frac{e^t}{e^t + 1}$$

More quantitatively, as can be seen from the analytical solution, the logistic curve shows early exponential growth for negative t, which

slows to linear growth of slope 1/4 near t = 0, then approaches y = 1 with an exponentially decaying gap.

The logistic equation is commonly applied as a model of population growth, where the rate of reproduction is proportional to both the existing population and the amount of available resources, all else being equal. The equation describes the self-limiting growth of a biological population.

Letting P represent population size (N is often used in ecology instead) and t represent time, this model is formalized by the differential equation:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

where the constant r defines the growth rate and K is the carrying capacity. In the equation, the early, unimpeded growth rate is modeled by the first term +rP. The value of the rate r represents the proportional increase of the population P in one unit of time. Later, as the population grows, the second term, which multiplied out is $-rP2/K$, becomes larger than the first as some members of the population P interfere with each other by competing for some critical resource, such as food or living space. This antagonistic effect is called the bottleneck, and is modeled by the value of the parameter K. The competition diminishes the combined growth rate, until the value of P ceases to grow (this is called maturity of the population).

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Linear Equations

Linear equations are equations of a single variable.

KEY POINTS

- Linear equations involve a single variable and an arbitrary number of constants.
- Linear equations are so-called because their most basic form is described by a line on a graph.
- Linear differential equations are differential equations which involve a single variable and its derivative.

A **linear equation** is an algebraic equation in which each term is either a constant or the product of a constant and a single variable.

A common form of a linear equation in the two variables x and y is $y = mx + b$ where m and b designate constants. The origin of the name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane. In this particular equation, the constant m determines the slope or gradient of that line, and the constant term b determines the point at which the line crosses the y-axis, otherwise known as the y-intercept.

Since terms of linear equations cannot contain products of distinct or equal variables, nor any power (other than 1) or other function of a variable, equations involving terms such as xy, x^2, y^1/3, and sin(x) are nonlinear.

Linear equations can be written in parametric form:

$x = Tt + U; y = Vt + W$

Two simultaneous equations in terms of a variable parameter t, with slope m = V/T, x-intercept (VU−WT)/V, and y-intercept (WT −VU)/T.

Linear differential equations are differential equations that have solutions which can be added together to form other solutions. They can be ordinary or partial. Linear differential equations are of the form $Ly = f$ where the differential operator L is a linear operator, y is the unknown function (such as a function of time y(t)), and f is a given function of the same nature as y (called the source term). For a function dependent on time, we may write the equation more expressly as $Ly(t) = f(t)$

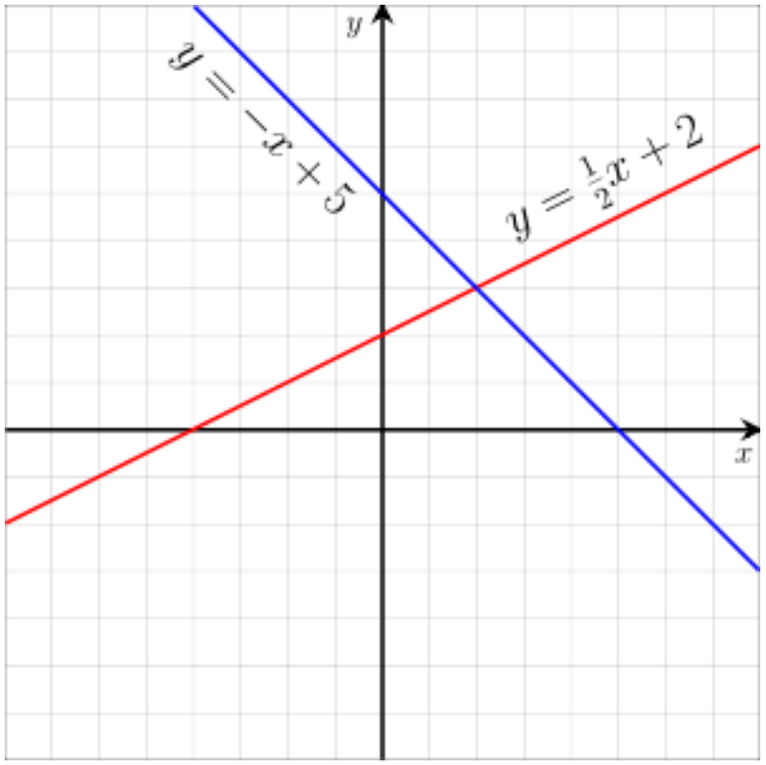

Figure 4.7 Linear equations

Graph example of linear equations

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Predator-Prey Systems

The relationship between predators and their prey can be modeled by a set of differential equations.

KEY POINTS

- The populations of predators and prey depend on each other.
- When there are many predators there are few prey. As the predators die off from lack of food, the prey population rebounds, enabling it to sustain a larger population of predators.
- This up and down cycle of populations can be well represented by differential equations and has a periodic solution.

The **predator–prey** equations are a pair of first-order, non-linear, **differential equations** frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey.

They evolve in time according to the pair of equations:

$$\frac{dx}{dt} = x(\alpha - \beta y); \frac{dy}{dt} = -y(\gamma - \delta y)$$

where x is the number of prey (for example, rabbits); y is the number of some predator (for example, foxes); and represent the growth rates of the two populations over time; t represents time; and α, β, γ and δ are parameters describing the interaction of the two species.

The model makes a number of assumptions about the environment and evolution of the predator and prey populations:

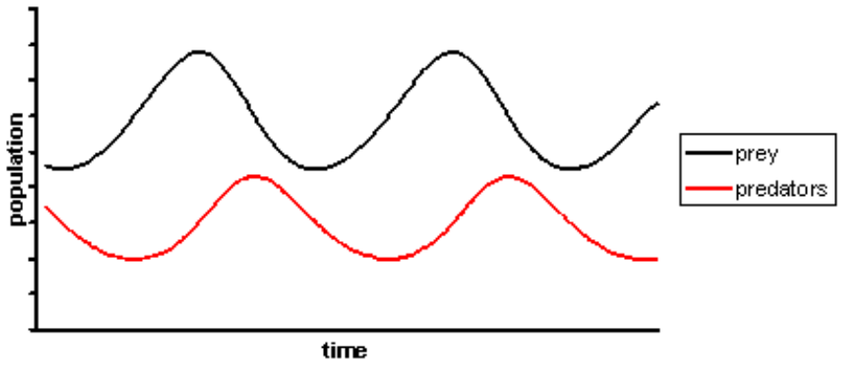
- The prey population finds ample food at all times.
- The food supply of the predator population depends entirely on the prey populations.
- The rate of change of population is proportional to its size.
- During the process, the environment does not change in favor of one species and the genetic adaptation is sufficiently slow.
- As differential equations are used, the solution is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping.

The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented in the equation above by the term αx. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented above by βxy. If either x or y is zero, then there can be no predation. With these two terms, the equation above can be interpreted as follows: the change in the prey's number is given by its own growth minus the rate at which it is preyed upon.

In the predator equation, δxy represents the growth of the predator population. (Note the similarity to the predation rate; however, a different constant is used as the rate at which the predator population grows is not necessarily equal to the rate at which it consumes the prey). γy represents the loss rate of the predators due to either natural death or emigration; it leads to an exponential decay in the absence of prey. Hence, the equation expresses the change in the predator population as growth fueled by the food supply, minus natural death.

The equations have periodic solutions and do not have a simple expression in terms of the usual trigonometric functions. They can only be solved numerically. However, a linearization of the equations yields a solution similar to simple harmonic motion with the population of predators following that of prey by 90°(*Figure 4.8*).

Figure 4.8 Solution to the equation



The solutions to the equations are periodic. The predator population follows the prey population.

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Section 2

Parametric Equations and Polar Coordinates

Parametric Equations

Calculus with Parametric Curves

Polar Coordinates

Area and Arc Length in Polar Coordinates

Conic Sections

Conic Sections in Polar Coordinates

Arc Length and Speed

https://www.boundless.com/calculus/differential-equations-parametric-equations-and-sequences-and-series/parametric-equations-and-polar-coordinates/

Parametric Equations

Parametric equations occur when the coordinates (ex: x and y) are expressed in terms of a single parameter.

KEY POINTS

- Parametric equations are useful for drawing curves, as the equation can be integrated and differentiated term-wise.
- A common example occurs in kinematics, where the trajectory of a point is usually represented by a parametric equation with the time as parameter.
- Equations can be converted between parametric equations and a single equation.

In mathematics, a parametric equation of a curve is a representation of the curve through equations expressing the **coordinates** of the points of the curve as functions of a variable called parameter. For example,

$x = cos(t); y = sin(t)$

is a parametric equation for the unit circle, where t is the parameter. A common example occurs in kinematics, where the trajectory of a point is usually represented by a parametric equation with time as the parameter. The notion of parametric equation has been generalized to surfaces of higher dimension with a number of parameters equal to the dimension of the manifold (dimension one and one parameter for curves, dimension two and two parameters for surfaces, etc.)

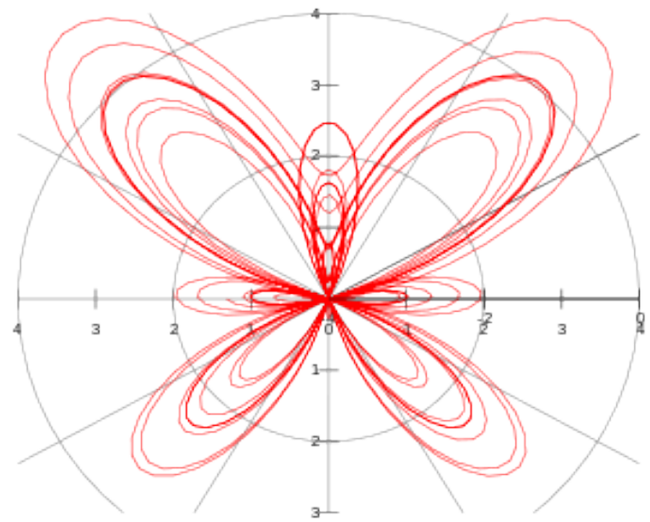

Figure 4.9
Parametric Example
One example of a sketch defined by parametric equations. Note that it is graphed on parametric axes.

For example, the simplest equation for a parabola $y = x^2$ can be parametrized by using a free parameter t, and setting $x = t, y = t^2$.

This way of expressing curves is practical as well as efficient; for example, one can integrate and differentiate such curves term-wise. Thus, one can describe the velocity of a particle following such a parametrized path as follows: $v(t) = (x'(t), y'(t))$, a function of the derivatives of x and y with respect to the parameter t.

Converting a set of parametric equations to a single equation involves eliminating the variable from the simultaneous equations. If one of these equations can be solved for t, the expression

obtained can be substituted into the other equation to obtain an equation involving x and y only. If there are rational functions, then the techniques of the theory of equations such as resultants can be used to eliminate t. In some cases there is no single equation in closed form that is equivalent to the parametric equations.

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Calculus with Parametric Curves

Calculus can be applied to parametric equations as well.

KEY POINTS

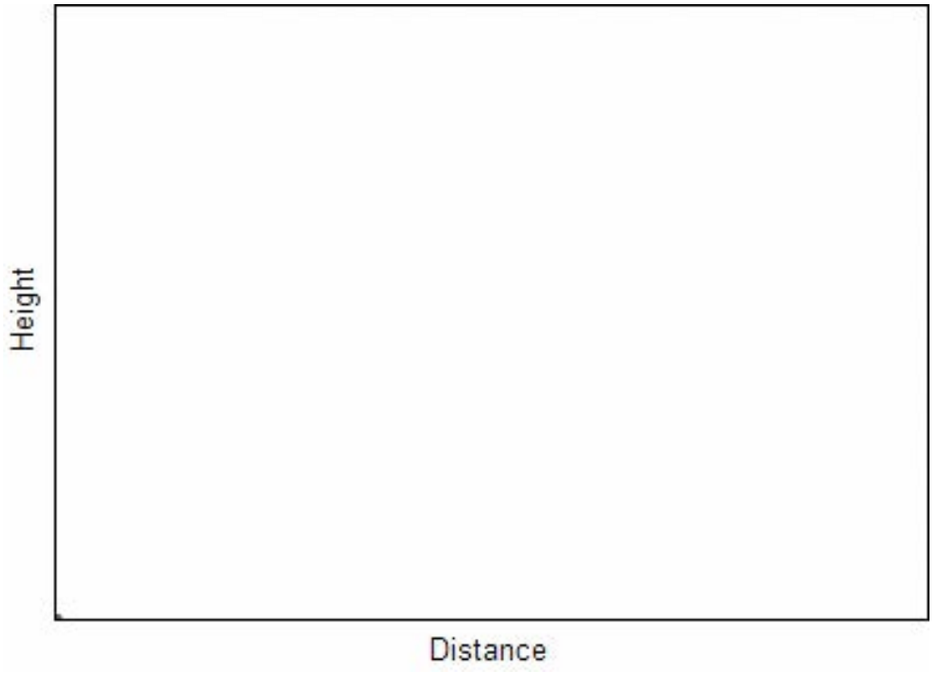
- Parametric equations are equations that depend on a single parameter.
- A common example comes from physics. The trajectory of an object is well represented by parametric equations.
- Writing the horizontal and vertical displacement in terms of the time parameter makes finding the velocity a simple matter of differentiating by the parameter time. Parameterizing makes this kind of analysis straight-forward.

Parametric equations are equations which depend on a single parameter. You can rewrite y=x such that x=t and y=t where t is the parameter.

A common example occurs in physics, where it is necessary to follow the **trajectory** of a moving object. The position of the object is given by x and y, signifying horizontal and vertical **displacement**, respectively. As time goes on the object flies

through its path and x and y change. Therefore, we can say that both x and y depend on a parameter t, which is time (*Figure 4.10*).

Figure 4.10 Trajectories



A trajectory is a useful place to use parametric equations because it relates the horizontal and vertical distance to the time.

This way of expressing curves is practical as well as efficient; for example, one can integrate and differentiate such curves term-wise. Thus, one can describe the velocity of a particle following such a parametrized path as:

$$v(t) = r'(t) = (x'(t), y'(t), z'(t)) = (-a\sin(t), a\cos(t), b)$$

where v is the velocity, r is the distance, and x, y, and z are the coordinates. The apostrophe represents the derivative with respect to the parameter.

The acceleration can be written as follows with the double apostrophe signifying the second derivative:

$$a(t) = r''(t) = (x''(t), y''(t), z''(t)) = (-a\cos(t), -a\sin(t), b)$$

Writing these equations in parametric form gives a common parameter for both equations to depend on. This makes integration and differentiation easier to carry out as they rely on the same variable. Writing x and y explicitly in terms of t enables one to differentiate and integrate with respect to t. The horizontal velocity is the time rate of change of the x value, and the vertical velocity is the time rate of change of the y value. Writing in parametric form makes this easier to do.

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Polar Coordinates

Polar coordinates define the location of an object in a plane by using a distance and an angle from a reference point and axis.

KEY POINTS

- Polar coordinates use a distance from a central point called a radial distance, usually specified as r.
- Polar coordinates use an angle measurement from a polar axis, which is usually positioned as horizontal and pointing to the right. Counterclockwise is usually positive.
- To convert from Polar coordinates to Cartesian coordinates, draw a triangle from the horizontal axis to the point. The x coordinate is r*cos(theta) and the y coordinate is r*sin(theta).

What is a Coordinate System

We use coordinate systems every day even if we don't realize it. For example, if you walk 20 meters to the right of the parking lot to find the car, you are using a coordinate system. Coordinate systems are a way of determining the location of a point or object of interest in relation to something else. The coordinate system you are most likely familiar with is the x, y coordinate system, where locations are described as horizontal (x) and vertical (y) distances from an arbitrary point. This is called the Cartesian coordinate system.

The x,y or Cartesian coordinate system is not always the easiest system to use for every problem. In certain problems, like those involving circles, it is easier to define the location of a point in terms of a distance and an angle. Such definitions are called **polar** coordinates.

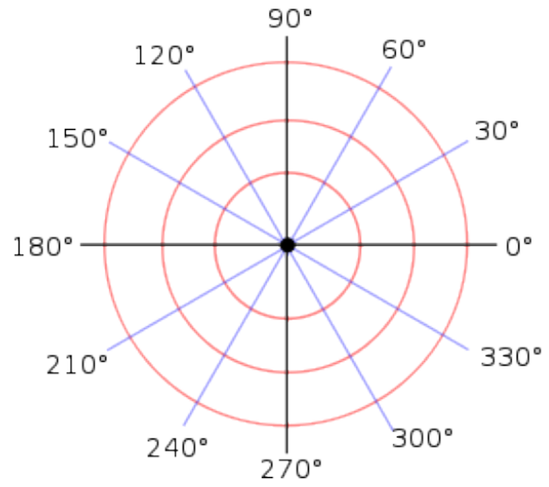
In polar coordinates, each point on a plane is defined by a distance from a fixed point and an angle from a fixed direction.

The distance is known as the radial distance and is usually denoted as r.

The angle is known as the polar angle, or radial angle, and is usually given as θ.

A positive angle is usually measured counterclockwise from the polar axis, and a positive radius is in the same direction as the angle. A negative radius would be opposite the direction of the

Figure 4.11 Polar Coordinates



A set of polar coordinates. Note the polar angle increases as you go counterclockwise around the circle with 0 degrees pointing horizontally to the right.

angle and a negative angle would be measured clockwise from the polar axis. The polar axis is usually drawn horizontal and pointing to the right (*Figure 4.11*).

Polar coordinates in r and θ can be converted to Cartesian coordinates x and y. This can be done by noting that the radial distance r and the polar angle θ can define a triangle with a horizontal length x and vertical length y (*Figure 4.12*). Thus, using trigonometry, it can be shown that the x coordinate is $rcos(\theta)$ and the y coordinate is $rsin(\theta)$.

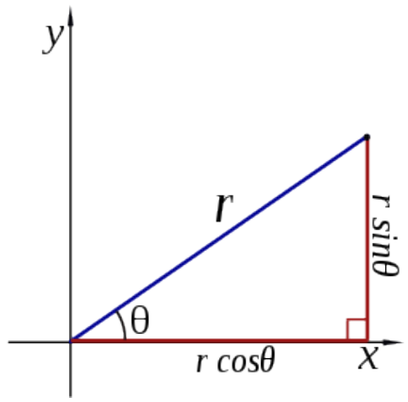


Figure 4.12
Relation between Cartesian and Polar Coordinates
The x Cartesian coordinate is given by r*cos(theta) and the y Cartesian coordinate is given by r*sin(theta).

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Area and Arc Length in Polar Coordinates

Area and arc length are calculated in polar coordinates by means of integration.

KEY POINTS

- Arc length is the linear length of the curve if it were straightened out.
- The area is the size of the region defined by the curve radius and the angle and length of the connection lines enclosing the area.
- To calculate these dimensions, use integration over the angle.

Arc Length

If you were to straighten a curved line out, the measured length would be the arc length. Since it can be very difficult to measure the length of an arc linearly, the solution is to use polar coordinates. Using polar coordinates allows us to integrate along the length of the arc in order to compute its length.

The arc length of the curve defined by a polar function is found by the integration over the curve r(θ). Let L denote this length along

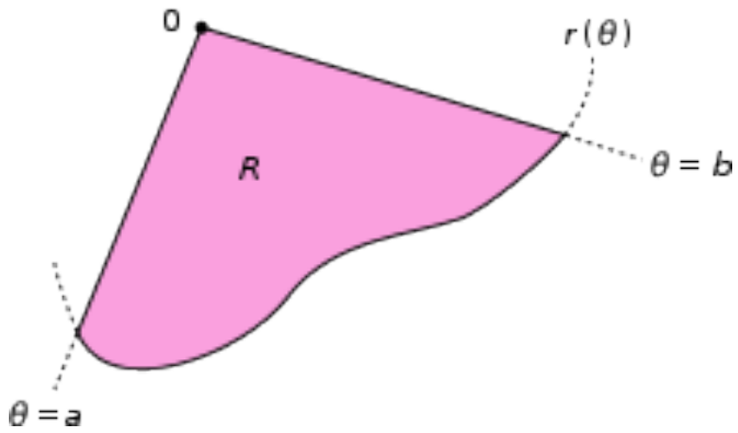

Figure 4.13 Arc Length

The curved lines bounding the region R are arcs. Their length can be calculated with calculus. The area of the region R can also be calculated by integration.

the curve starting from points A through to point B, where these points correspond to θ = a and θ = b such that $0 < b - a < 2\pi$. (*Figure 4.13*). The length of L is given by the following integral:

$$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$$

Solving this integral will give the length of the arc.

Arc Segment Area

To find the area enclosed by the arcs and the radius and polar angles, you again use integration.

Let R denote the region enclosed by a curve r(θ) and the rays θ = a and θ = b, where $0 < b - a \leq 2\pi$ (*Figure 4.13*). Then, the area of R is

$$1/2 \int_a^b r^2 d\theta$$

This result can be found as follows: First, the interval [a, b] is divided into n subintervals, where n is an arbitrary positive integer. Thus Δθ, the length of each subinterval, is equal to b – a (the total length of the interval), divided by n, the number of subintervals. For each subinterval i = 1, 2, ..., n, let θi be the midpoint of the subinterval, and construct a sector with the center at the pole, radius r(θi), central angle Δθ and arc length r(θi)Δθ. The area of each constructed sector is therefore equal to

$$1/2 r^2 \Delta\theta$$

And the total area is the sum of these sectors. An infinite sum of these sectors is the same as integration.

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Conic Sections

Conic sections are defined by intersections of cones with planes.

KEY POINTS

- Conic sections are curves obtained from an intersection of a cone with a plane.
- Graphs of quadratic equations in two variables are conic sections.
- Conic sections are important in astronomy: the orbits of two massive objects that interact according to Newton's law of universal gravitation are conic sections if their common center of mass is considered to be at rest.

In mathematics, a conic section (or just "conic") is a curve obtained from the intersection of a **cone** (more precisely, a right circular conical surface) with a plane. In analytic geometry, a conic may be defined as a plane algebraic curve of degree. There are a number of other geometric definitions possible, one of the most useful being that a conic consists of those points whose distances to some other point (called a focus) and some other line (called a directrix) are in a fixed ratio, called the eccentricity.

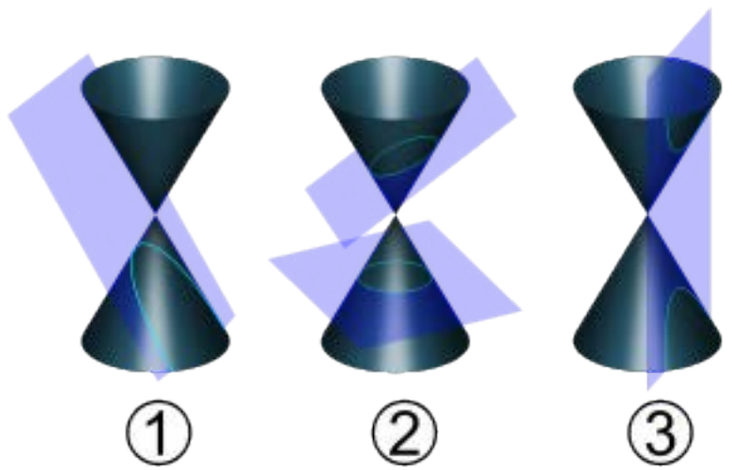

Figure 4.14 Types of Conic Sections
There are three types of conic sections:
1.Parabola;
2. Ellipse;
3. Hyperbola.

Traditionally, the three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, and is of such interest in its own right that it is sometimes called the fourth type of conic section. The type of a conic corresponds to its eccentricity-- those with eccentricity less than 1 being ellipses, those with eccentricity equal to 1 being parabolas, and those with eccentricity greater than 1 being hyperbolas. In the focus-directrix definition of a conic, the circle is a limiting case with eccentricity 0. In modern geometry, certain degenerate cases-- such as the union of two lines-- are included as conics as well.

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section-- though it may be degenerate-- and all conic sections arise in this way

Conic sections are important in astronomy: the orbits of two massive objects that interact according to Newton's law of universal

gravitation are conic sections if their common center of mass is considered to be at rest. If they are bound together, they will both trace out ellipses; if they are moving apart, they will both follow parabolas or hyperbolas. In projective geometry, the conic sections in the projective plane are equivalent to each other up to projective transformations.

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Conic Sections in Polar Coordinates

Conic sections are sections of cones and can be represented by polar coordinates.

KEY POINTS

- Conic sections are the intersections of cones with a plane.
- The three types of conic sections are the hyperbola, parabola, and ellipse.
- Polar coordinates offer us a useful way of representing conic sections.

In mathematics, a conic section (or just conic) is a curve obtained as the intersection of a cone (more precisely, a right circular conical surface) with a plane. There are a number of other geometric definitions possible. One of the most useful definitions, in that it involves only the plane, is that a conic consists of those points whose distances to some point-- called a focus-- and some line-- called a directrix-- are in a fixed ratio, called the eccentricity (*Figure 4.15*).

Traditionally, the three types of conic section are the **hyperbola**, the parabola, and the ellipse. The circle is a special case of the

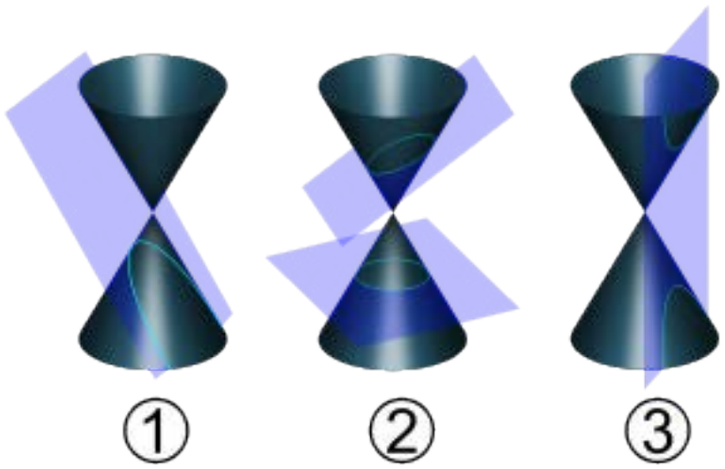

Figure 4.15 Conic Sections

Types of conic sections

ellipse, and is of such sufficient interest in its own right that it is sometimes called the fourth type of conic section. The type of a conic corresponds to its eccentricity, those with eccentricity less than 1 being ellipses, those with eccentricity equal to 1 being parabolas, and those with eccentricity greater than 1 being hyperbolas. In the focus-directrix definition of a conic, the circle is a limiting case with eccentricity 0. In modern geometry, certain degenerate cases, such as the union of two lines, are included as conics as well.

In polar coordinates, a conic section with one focus at the origin is given by the equation

$$r = \frac{l}{1 + e\cos(\theta)},$$

where e is the eccentricity and l is half the latus rectum (*Figure 4.16*). As in the figure, for e = 0, we have a circle, for 0 < e < 1 we obtain an ellipse, for e = 1 a parabola, and for e > 1 a hyperbola.

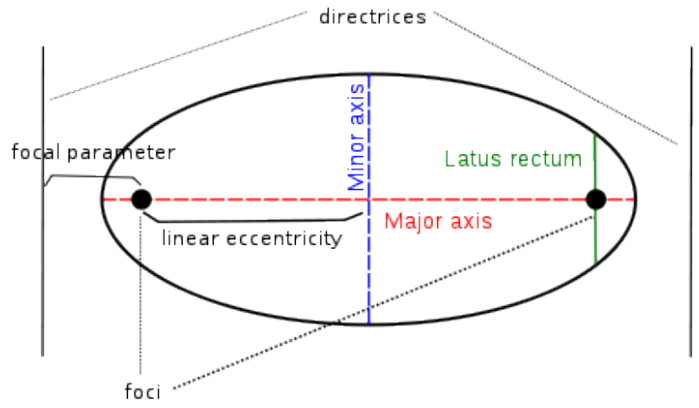

Figure 4.16 Conic Parameters

Conic parameters in the case of an ellipse

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Arc Length and Speed

Arc length and speed in parametric equations can be calculated using integration and the Pythagorean theorem.

KEY POINTS

- Arc length is the length of a curve. To calculate it in parametric equations, employ the Pythagorean Theorem.
- Arc lengths can be calculated by adding up a series of infinitesimal lengths along the arc. To do this, set up an integral over the parameter.
- Speed is the rate of change of the arc length with respect to time. The derivatives of x and y with respect to time are plugged into the Pythagorean Theorem to give the speed of an object traveling in an arc.

The length of a **curve** can be difficult to measure. A curve may be thought of as an infinite number of infinitesimal straight line segments, each pointing in a slightly different direction to make up the curve. Adding up all these lengths together would be equivalent to stretching the curve out straight and measuring its length. The length of the curve is called the arc length.

In order to calculate the arc length, we use integration because it is an efficient way to add up a series of infinitesimal lengths.

Arc lengths can be used to find the distance traveled by an object with an arcing path. Consider a case in which an object movies along a path in the Cartesian plane (the x, y plane). Its position horizontally is given by x=f(t) and its position vertically is given by y=g(t), where f and g are functions which depend on a parameter, t. Since there are two functions for position, and they both depend on a single parameter, time, we call these equations parametric equations.

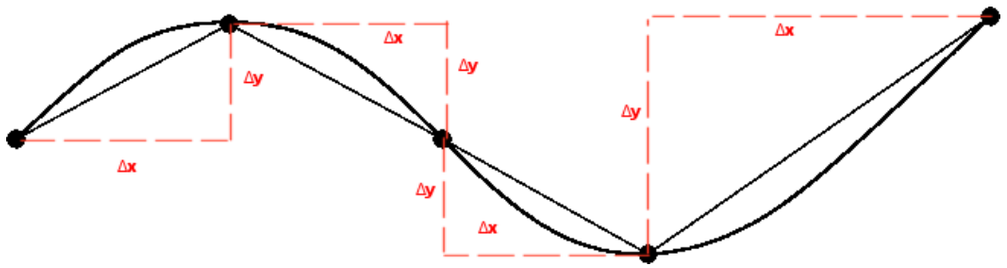
The distance, or arc length, the object travels through its motion is given by the equation

$$D = \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

This equation is obtained using the Pythagorean Theorem (*Figure 4.17*). The arc length is calculated by laying out an infinite number of infinitesimal right triangles along the curve. Each of these triangles has a width dx and a height dy, standing for an infinitesimal increase in x and y. By the Pythagorean Theorem, each hypotenuse will have length $\sqrt{dx^2 + dy^2}$. Adding up each tiny hypotenuse yields the arc length.

However, since x and y depend on the parameter t, we will want to integrate over t, not over x and y. Taking the derivative of x and y

Figure 4.17 Approximating Arc Length with Hypotenuses



The length of a curve can be approximated by using a series of right triangles with the hypotenuses lying along the curve. The smaller the triangles one uses, the closer the approximation will be.

with respect to t, we find the rate of change of the distance with time. This is also known as the speed. As shown previously using the Pythagorean Theorem, it is given by $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$, where the rate of change of the hypotenuse length depends on the rate of change of x and y. Integrating the speed with respect to time gives the distance as shown above.

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Section 3

Infinite Sequences and Series

Sequences

Series

The Integral Test and Estimates of Sums

Comparison Tests

Alternating Series

Absolute Convergence and Ratio and Root Tests

Tips for Testing Series

Power Series

Expressing Functions as Power Functions

Taylor and Maclaurin Series

Applications of Taylor Series

Summing an Infinite Series

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Sequences

A sequence is an ordered list of objects and can be considered as a function whose domain is the natural numbers.

KEY POINTS

- Like a set, a sequence contains members (also called elements). Unlike a set, order matters in a sequence, and the same elements can appear multiple times at different positions.
- The terms of a sequence are commonly denoted by a single variable, say a_n, where the index n indicates the n-th element of the sequence.
- Sequences whose elements are related to the previous elements in a straightforward way are often specified using recursion.

A sequence is an ordered list of objects (or events). Like a **set**, it contains members (also called elements, or terms). The number of ordered elements (possibly infinite) is called the length of the sequence. Unlike a set, order matters in a sequence, and exactly the same elements can appear multiple times at different positions in the sequence. Most precisely, a sequence can be defined as a function whose domain is a countable, totally ordered set, such as the natural numbers.

Indexing

The terms of a sequence are commonly denoted by a single variable, say a_n, where the index n indicates the nth element of the sequence. Indexing notation is used to refer to a sequence in the abstract. It is also a natural notation for sequences whose elements are related to the index n (the element's position) in a simple way (*Figure 4.18*). For instance, the sequence of the first 10 square numbers could be written as $(a_1, a_2, \ldots, a_{10})$, $a_k = k^2$. This represents the sequence (1,4,9,...100). Sequences can be indexed beginning and ending from any integer. The infinity symbol ∞ is often used as the superscript to indicate the sequence including all integer k-values starting with a

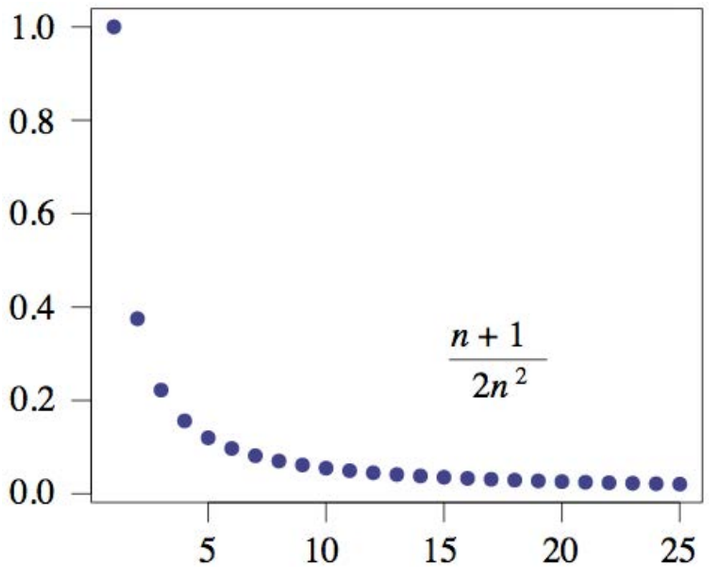

Figure 4.18 A Convergent Sequence

The plot of a convergent sequence (a_n) is shown in blue. Visually, we can see that the sequence is converging to the limit zero as n increases.

certain one. The sequence of all positive squares is then denoted as $(a_k)_{k=1}^{\infty}$, $a_k = k^2$.

Specifying a Sequence by Recursion

Sequences whose elements are related to the previous elements in a straightforward way are often specified using recursion. This is in contrast to the specification of sequence elements in terms of their position. To specify a sequence by recursion requires a rule to construct each consecutive element in terms of the ones before it. In addition, enough initial elements must be specified so that new elements of the sequence can be specified by the rule.

EXAMPLES

(M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y). Also, the sequence (1, 1, 2, 3, 5, 8), which contains the number 1 at two different positions, is a valid sequence. Sequences can be finite, as in this example, or infinite, such as the sequence of all even positive integers (2, 4, 6,...). Finite sequences are sometimes known as strings or words, and infinite sequences as streams. The empty sequence () is included in most notions of sequence, but may be excluded depending on the context.

The Fibonacci sequence can be defined using a recursive rule along with two initial elements. The rule is that each element is the sum of the previous two elements, and the first two elements are 0 and 1: $a_n = a_{n-1} + a_{n-2}$ and $a_0 = 0, a_1 = 1$. The first ten terms of this sequence are 0,1,1,2,3,5,8,13,21, and 34.

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Series

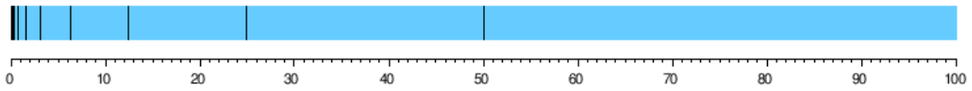
A series is the sum of the terms of a sequence.

KEY POINTS

- Given an infinite sequence of numbers $\{ a_n \}$, a series is informally the result of adding all those terms together: $\sum_{n=0}^{\infty} a_n$.
- Unlike finite summations, infinite series need tools from mathematical analysis, specifically the notion of limits, to be fully understood and manipulated.
- By definition, a series converges to a limit L if and only if the associated sequence of partial sums converges to L: $L = \sum_{n=0}^{\infty} a_n \Leftrightarrow L = \lim_{k \to \infty} S_k$.

A series is, informally speaking, the sum of the terms of a **sequence**. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Given an infinite sequence of numbers $\{ a_n \}$, a series is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + \cdots$. These can be written more compactly using the summation symbol $\sum$. An example is the famous series from **Zeno's dichotomy** (*Figure 4.19*) and its mathematical representation:

Figure 4.19 Zeno's Paradox



Say you are working from a location x=0 toward x=100. Before you can get there, you must get halfway there. Before you can get halfway there, you must get a quarter of the way there. Before traveling a quarter, you must travel one-eighth; before an eighth, one-sixteenth; and so on.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

The terms of the series are often produced according to a certain rule, such as by a formula or by an algorithm. As there are an infinite number of terms, this notion is often called an infinite series. Unlike finite summations, infinite series need tools from mathematical analysis, specifically the notion of limits, to be fully understood and manipulated. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, and finance.

Definition

For any sequence of rational numbers, real numbers, complex numbers, functions thereof, etc., the associated series is defined as

the ordered formal sum: $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$. The sequence of

partial sums $\{S_k\}$ associated to a series $\sum_{n=0}^{\infty} a_n$ is defined for each k as

the sum of the sequence $\{a_n\}$ from a_0 to a_k.

$$S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k$$

By definition, the series $\sum_{n=0}^{\infty} a_n$ converges to a limit L if and only if

the associated sequence of partial sums $\{S_k\}$ converges to L. This

definition is usually written as $L = \sum_{n=0}^{\infty} a_n \Leftrightarrow L = \lim_{k\to\infty} S_k$.

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The Integral Test and Estimates of Sums

The integral test is a method of testing infinite series of nonnegative terms for convergence by comparing them to an improper integral.

KEY POINTS

- The integral test uses a monotonically decreasing function f defined on the unbounded interval [N, ∞) (N:integer).
- The infinite series $\int_N^{\infty} f(x)\,dx$ is finite. In other words, if the integral diverges, then the series diverges as well.
- Integral tests proves that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The integral test for convergence is a method used to test infinite series of non-negative terms for convergence. It was developed by Colin Maclaurin and Augustin-Louis Cauchy and is sometimes known as the Maclaurin–Cauchy test.

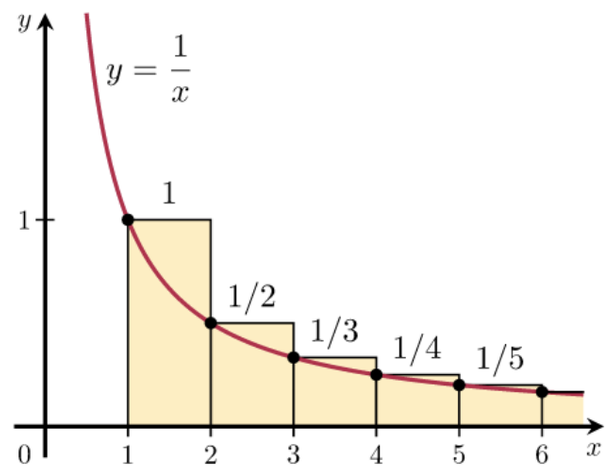
Statement of the test

Consider an integer N and a non-negative function f defined on the unbounded interval [N, ∞), on which it is monotonically decreasing.

The infinite series $\sum_{n=N}^{\infty} f(n)$ converges to a real number if and only if the improper integral $\int_N^{\infty} f(x)\,dx$ is finite. In other words, if the integral diverges, then the series diverges as well.

Although we won't go into the details, the proof of the test also gives the lower and upper bounds $\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} f(n) \le f(N) + \int_N^{\infty} f(x)\,dx$ for the infinite series.

Figure 4.20 Integral Test



The integral test applied to the harmonic series. Since the area under the curve y = 1/x for x ∈ [1, ∞) is infinite, the total area of the rectangles must be infinite as well.

Applications

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because, using the natural logarithm (its derivative) and the fundamental theorem of calculus, we get $\int_1^M \frac{1}{x}\,dx = \ln x \Big|_1^M = \ln M \to \infty \quad \text{for } M \to \infty$ (*Figure 4.20*).

On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$ converges for every ε > 0 because, by the power rule,

$$\int_1^M \frac{1}{x^{1+\varepsilon}}\,dx = -\frac{1}{\varepsilon x^{\varepsilon}}\Big|_1^M = \frac{1}{\varepsilon}\left(1 - \frac{1}{M^{\varepsilon}}\right) \le \frac{1}{\varepsilon} < \infty \quad \text{for all } M \ge 1.$$

The above examples involving the harmonic series raise the question of whether there are monotone sequences such that f(n) decreases to 0 faster than 1/n but slower than $1/n^{1+\varepsilon}$ in the sense that $\lim_{n\to\infty} \frac{f(n)}{1/n} = 0$ and $\lim_{n\to\infty} \frac{f(n)}{1/n^{1+\varepsilon}} = \infty$ for every ε > 0, and whether the corresponding series of the f(n) still diverges. Once such a sequence is found, a similar question can be asked of f(n) taking the role of 1/n and so on. In this way, it is possible to investigate the borderline between divergence and convergence of infinite series.

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Comparison Tests

Comparison test may mean either limit comparison test or direct comparison test, both of which can be used to test convergence of a series.

KEY POINTS

- For sequences $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ (0).
- If the infinite series $\sum a_n$ also converges.
- If the infinite series $\sum a_n$ also diverges.

Comparison tests may mean either limit comparison tests or direct comparison tests. The limit comparison test is a method of testing for the convergence of an infinite series, while the direct comparison test is a way of deducing the convergence or divergence of an infinite series or an **improper integral** by comparison with other series or integral whose convergence properties are already known.

Limit Comparison Test

Statement: Suppose that we have two series, $\Sigma_n a_n$ and $\Sigma_n b_n$, where a_n, b_n are greater than or equal to 0 for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then either both series converge or both series diverge.

Example: We want to determine if the series $\Sigma\frac{n+1}{2n^2}$ converges or diverges. For this we compare it with the series $\Sigma\frac{1}{n}$, which diverges (see the atom on integral tests). As $\lim_{n\to\infty} \frac{n+1}{2n^2}\frac{n}{1} = \frac{1}{2}$, we have that the original series also diverges (*Figure 4.21*).

Direct Comparison Test

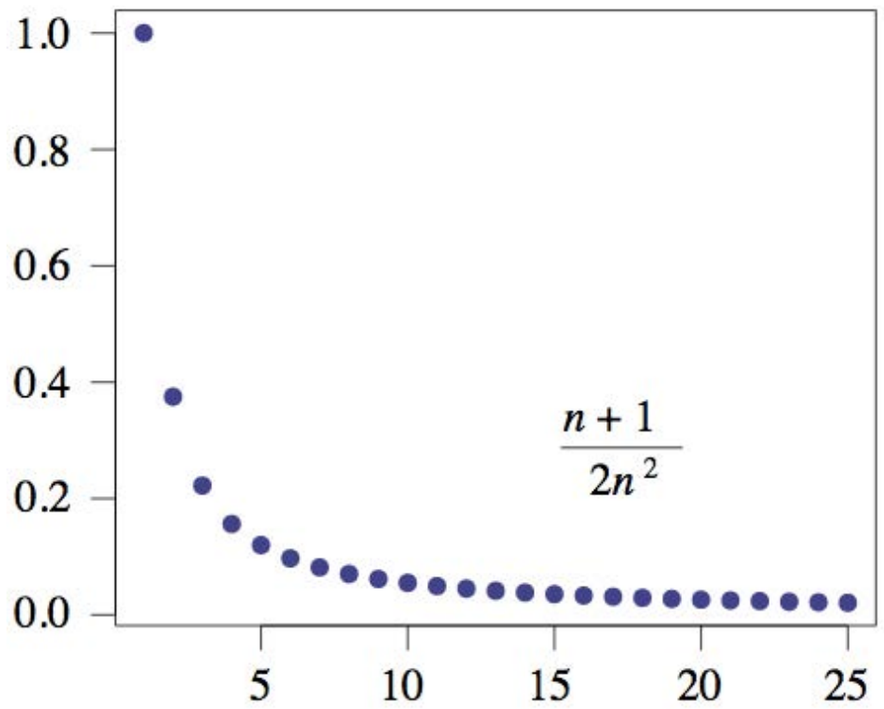
The direct comparison test provides a way of deducing the convergence or divergence of an infinite series or an improper integral. In both cases, the test works by comparing the given series or integral to one whose convergence properties are known. In this atom, we will check the series case only.

For sequences $\{a_n\}, \{b_n\}$ with non-negative terms,

- If the infinite series $\sum b_n$ converges and $0 \le a_n \le b_n$ for all sufficiently large n (that is, for all n>N for some fixed value N), then the infinite series $\sum a_n$ also converges.
- If the infinite series $\sum b_n$ diverges and $a_n \ge b_n \ge 0$ for all sufficiently large n, then the infinite series $\sum a_n$ also diverges.

Example: The series $\Sigma\frac{1}{n^3+2n}$ converges because $\frac{1}{n^3+2n} < \frac{1}{n^3}$ for (n>0) and $\Sigma\frac{1}{n^3}$ converges (see the atom on **integral tests**).

Figure 4.21 Limit Convergence Test



The ratio between $(n+1)/2n^2$ and 1/n for n->∞ is 1/2. Since the sum of the sequence $1/n(\sum 1/n)$ diverges, the limit convergence test tells that the original series (with $(n+1)/2n^2$) also diverges.

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Alternating Series

An alternating series is an infinite series of the form $\sum_{n=0}^{\infty}(-1)^n a_n$ or $\sum_{n=0}^{\infty}(-1)^{n-1} a_n$ with an > 0 for all n.

KEY POINTS

- The theorem known as the "Leibniz Test," or the alternating series test, tells us that an alternating series will converge if the terms an converge to 0 monotonically.
- The signs of the general terms alternate between positive and negative.
- The sum $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

An alternating series is an infinite series of the form $\sum_{n=0}^{\infty}(-1)^n a_n$ or $\sum_{n=0}^{\infty}(-1)^{n-1} a_n$ with $a_n > 0$ for all n. The signs of the general terms alternate between positive and negative. Like any series, an alternating series converges if and only if the associated sequence of partial sums converges.

Alternating Series Test

The theorem known as the "Leibniz Test," or the alternating series test, tells us that an alternating series will converge if the terms a_n converge to 0 monotonically.

Proof: Suppose the sequence a_n converges to zero and is monotone decreasing. If m is odd and $S_m - S_n < a_m$ via the following calculation:

$$S_m - S_n = \sum_{k=0}^{m} (-1)^k a_k - \sum_{k=0}^{n} (-1)^k a_k = \sum_{k=m+1}^{n} (-1)^k a_k$$

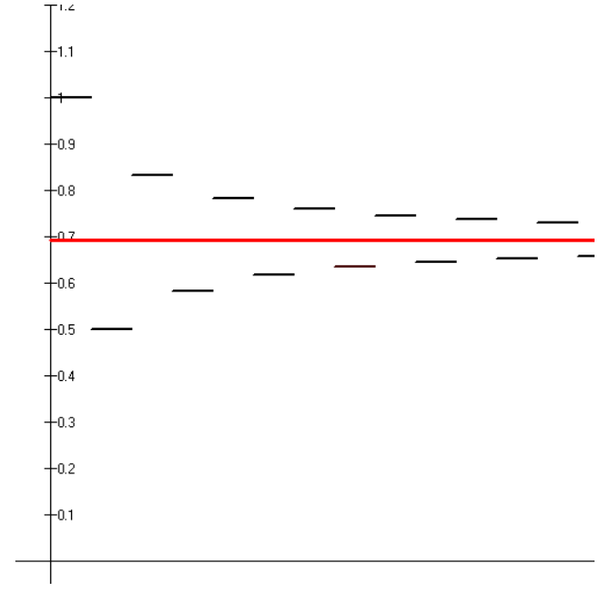
$$= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$$

$$= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots - a_n \leq a_{m+1} \leq a_m \, [a_n decreasing] \, .$$

Since a_n is monotonically decreasing, the terms $-(a_m - a_{m+1})$ are negative. Thus, we have the final inequality $S_m - S_n \leq a_m$. Similarly, it can be shown that, since a_m converges to 0, S_m - S_n converges to 0 for $m, n \to \infty$. Therefore, our partial sum S_m converges. (The sequence $\{S_m\}$ is said to form a Cauchy sequence, meaning that elements of the sequence become arbitrarily close to each other as the sequence progresses.) The argument for m even is similar.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (*Figure 4.22*).

Figure 4.22 Alternating Harmonic Series



The first fourteen partial sums of the alternating harmonic series (black line segments) shown converging to the natural logarithm of 2 (red line).

$a_n = \frac{1}{n}$ converges to 0 monotonically. Therefore, the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

EXAMPLE

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

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Absolute Convergence and Ratio and Root Tests

An infinite series of numbers is said to converge absolutely if the sum of the absolute value of the summand is finite.

KEY POINTS

- A real or complex series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=0}^{\infty} \left|a_n\right| = L$ for some real number L.
- The root test is a convergence test of an infinite series that makes use of the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.
- The root test is a criterion for the convergence of an infinite series using the limit superior $C = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

An infinite series of numbers is said to converge absolutely (or to be absolutely convergent) if the sum of the absolute value of the **summand** is finite. More precisely, a real or complex series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=0}^{\infty} \left|a_n\right| = L$ for some real number L. Similarly, an improper integral of a function, $\int_0^{\infty} f(x)\,dx$,

is said to converge absolutely if the integral of the absolute value of the integrand is finite—that is, if $\int_0^\infty \left|f(x)\right| dx = L$.

Absolute convergence is important for the study of infinite series because its definition is strong enough to have properties of finite sums that not all convergent series possess, yet is broad enough to occur commonly. (A convergent series that is not absolutely convergent is called conditionally convergent.)

Ratio Test

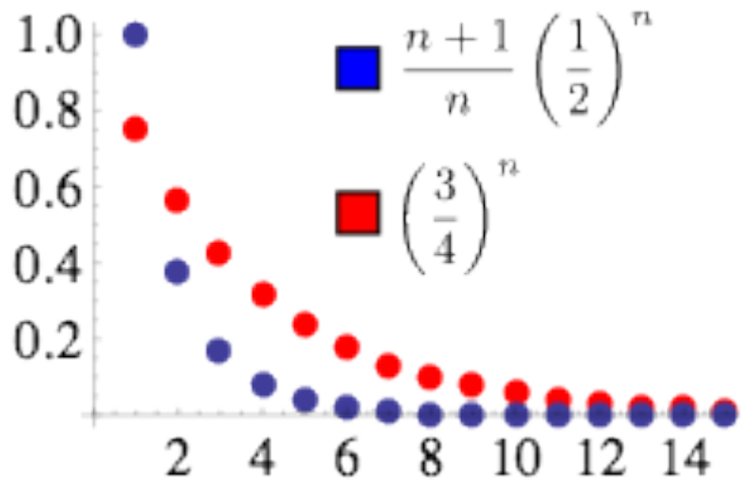
The ratio test is a test (or "criterion") for the convergence of a series $\sum_{n=1}^{\infty} a_n$, where each term is a real or complex number and a_n is nonzero when n is large. The test was first published by Jean le Rond d'Alembert and is sometimes known as d'Alembert's ratio test.

The usual form of the test makes use of the limit, $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

The ratio test states that,

- if L < 1, then the series converges absolutely (*Figure 4.23*);
- if L > 1, then the series does not converge;
- if L = 1 or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

Figure 4.23 Ratio Test



In this example, the ratio of adjacent terms in the blue sequence converges to L=1/2. We choose r = (L+1)/2 = 3/4. Then the blue sequence is dominated by the red sequence rk for all n ≥ 2. The red sequence converges, so the blue sequence does as well.

Root Test

The root test is a criterion for the convergence (a convergence test) of an infinite series. It depends on the quantity $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$,

where a_n are the terms of the series, and states that the series converges absolutely if this quantity is less than one but diverges if it is greater than one. It is particularly useful in connection with power series.

The root test was developed first by Augustin-Louis Cauchy and so is sometimes known as the Cauchy root test, or Cauchy's radical test. For a series $\sum_{n=1}^{\infty} a_n$, the root test uses the number

$C = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, where "lim sup" denotes the limit superior, possibly ∞. Note that if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ converges, then it equals C and may be used in the root test instead. The root test states that

- if C < 1, then the series converges absolutely;
- if C > 1, then the series diverges;
- if C = 1 and the limit approaches strictly from above, then the series diverges;
- otherwise the test is inconclusive (the series may diverge, converge absolutely, or converge conditionally).

There are some series for which C = 1 and the series converges, e.g. $\sum 1/n^2$, and there are others for which C = 1 and the series diverges, e.g. $\sum 1/n$.

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Tips for Testing Series

Convergence tests are methods of testing for the convergence or divergence of an infinite series.

KEY POINTS

- There is no single convergence test which works for all series out there.
- Practice and training will help you choose the right test for a given series.
- We have learned about the root/ratio test, integral test, and direct/limit comparison test.

Convergence tests are methods of testing for the convergence, **conditional convergence**, absolute convergence, interval of convergence, or divergence of an infinite series. When testing the convergence of a series, you should remember that there is no single convergence test which works for all series. It is up to you to guess and pick the right test for a given series. Practice and training will help you in expediting this "guessing" process.

Here is a summary for the convergence test that we have learned:

List of Tests

Limit of the Summand: If the limit of the summand is undefined or nonzero, then the series must diverge.

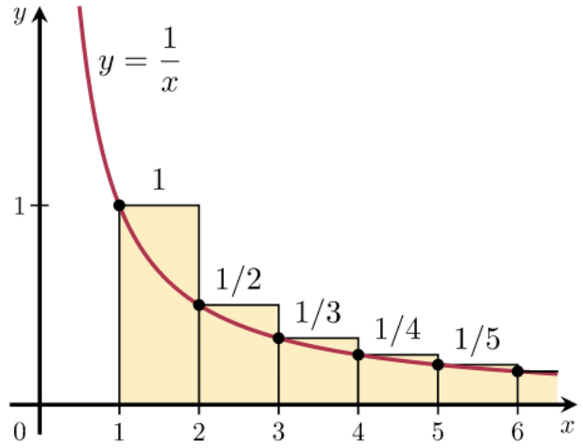
Ratio test: For $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, if r<1, the series converges; if r>1, the series diverges; if r=1, the test is inconclusive.

Root test: For $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, if r < 1, then the series converges; if r > 1, then the series diverges; if r = 1, the root test is inconclusive.

Integral test: For a positive, monotone decreasing function f(x) such that f(n) = a_n, if $\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx < \infty$ then the series converges. But if the integral diverges, then the series does so as well (*Figure 4.24*).

Direct comparison test: If the series $\sum_{n=1}^{\infty} b_n$ is an absolutely convergent series and $|a_n| \leq |b_n|$ for sufficiently large n, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Figure 4.24 Integral Test



The integral test applied to the harmonic series. Since the area under the curve y = 1/x for x ∈ [1, ∞) is infinite, the total area of the rectangles must be infinite as well.

Limit comparison test: If $\{a_n\}, \{b_n\} > 0$, and the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is not zero, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

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Power Series

A power series (in one variable) is an infinite series of the form $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, where a_n is the coefficient of the n-th term and x varies around c.

KEY POINTS

- Power series usually arise as the Taylor series of some known function.
- In many situations c is equal to zero-- for instance, when considering a Maclaurin series. In such cases, the power series takes the simpler form $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$
- A power series will converge for some values of the variable x and may diverge for others. If there exists a number r with $0 < r \leq \infty$ such that the series converges when $|x - c| < r$ and diverges when $|x - c| > r$, the number r is called the radius of convergence of the power series.

A power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

where a_n represents the coefficient of the n-th term, c is a constant,

and x varies around c (for this reason one sometimes speaks of the series as being centered at c). This series usually arises as the Taylor series of some known function (*Figure 4.25*). Any polynomial can be easily expressed as a power series around any center c, albeit one with most coefficients equal to zero. For instance, the polynomial $f(x) = x^2 + 2x + 3$ can be written as a power series around the center c=1 as $f(x) = 6 + 4(x-1) + 1(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + \cdots$, or, indeed, around any other center c.

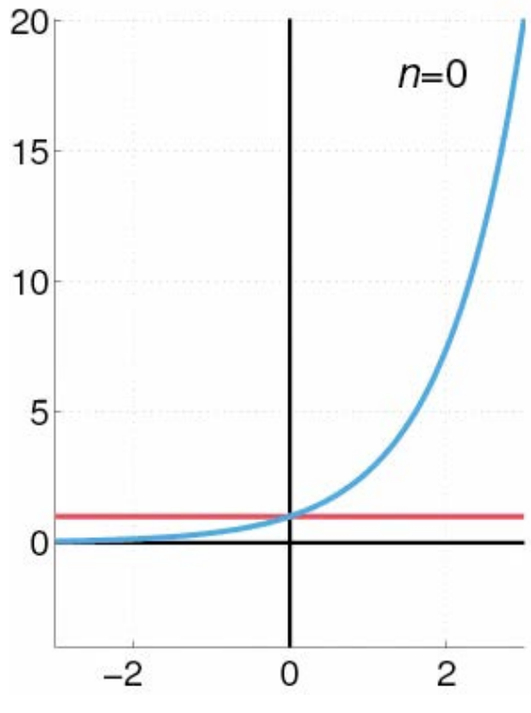

Figure 4.25
Exponential Function as a Power Series
The exponential function (in blue), and the sum of the first n+1 terms of its Maclaurin power series (in red).

In many situations c is equal to zero-- for instance, when considering a Maclaurin series. In such cases, the power series takes the simpler form $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. These power series arise primarily in real and complex analysis, but also occur in combinatorics (under the name of generating functions) and in electrical engineering (under the name of the Z-transform). The familiar decimal notation for real numbers can also be viewed as an example of a power series, with integer coefficients, but with the argument x fixed at 1/10. In number theory, the concept of p-adic numbers is also closely related to that of a power series.

Radius of Convergence

A power series will converge for some values of the variable x and may diverge for others. All power series f(x) in powers of (x-c) will converge at x = c. If c is not the only convergent point, then there is always a number r with $0 < r \leq \infty$ such that the series converges whenever $|x - c| < r$ and diverges whenever $|x - c| > r$. The number r is called the radius of convergence of the power series. According to the Cauchy-Hadamard theorem, the radius r can be computed from $r^{-1} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, if this limit exists.

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Expressing Functions as Power Functions

A power function is a function of the form $f(x) = cx^r$ where c and r are constant real numbers.

KEY POINTS

- Since all infinitely differentiable functions can be represented in power series, any infinitely differentiable function can be represented as a sum of many power functions (of integer exponents).
- Therefore, an arbitrary function that is infinitely differentiable is expressed as an infinite sum of power functions (x^n) of integer exponent: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
- Functions of the form f(x) = x3, f(x) = x1.2, f(x) = x-4 are all power functions.

A power function is a function of the form $f(x) = cx^r$ where c and r are constant real numbers. Polynomials are made of power functions. Since all infinitely **differentiable** functions can be represented in power series, any infinitely differentiable function can be represented as a sum of many power functions (of integer exponents). The domain of a power function can sometimes be all real numbers, but generally a non-negative value is used to avoid problems with simplifying. The domain of definition is determined by each individual case. Power functions are a special case of **power law** relationships, which appear throughout mathematics and statistics.

The Taylor series of a real or complex-valued function f(x) that is infinitely differentiable in a neighborhood of a real or complex number a is the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, where n! denotes the factorial of n and $f^{(n)}$(a) denotes the n-th derivative of f evaluated at the point x=a. Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial (*Figure 4.26*). If the Taylor series is centered at zero, then that series is also called a Maclaurin series, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Therefore, an arbitrary function that is infinitely differentiable is expressed as an infinite sum of power functions (x^n) of integer exponent.

Examples

Functions of the form f(x) = x^3, f(x) = $x^{1.2}$, f(x) = x^{-4} are all power functions.

Figure 4.26 sin(x) in Taylor Approximations

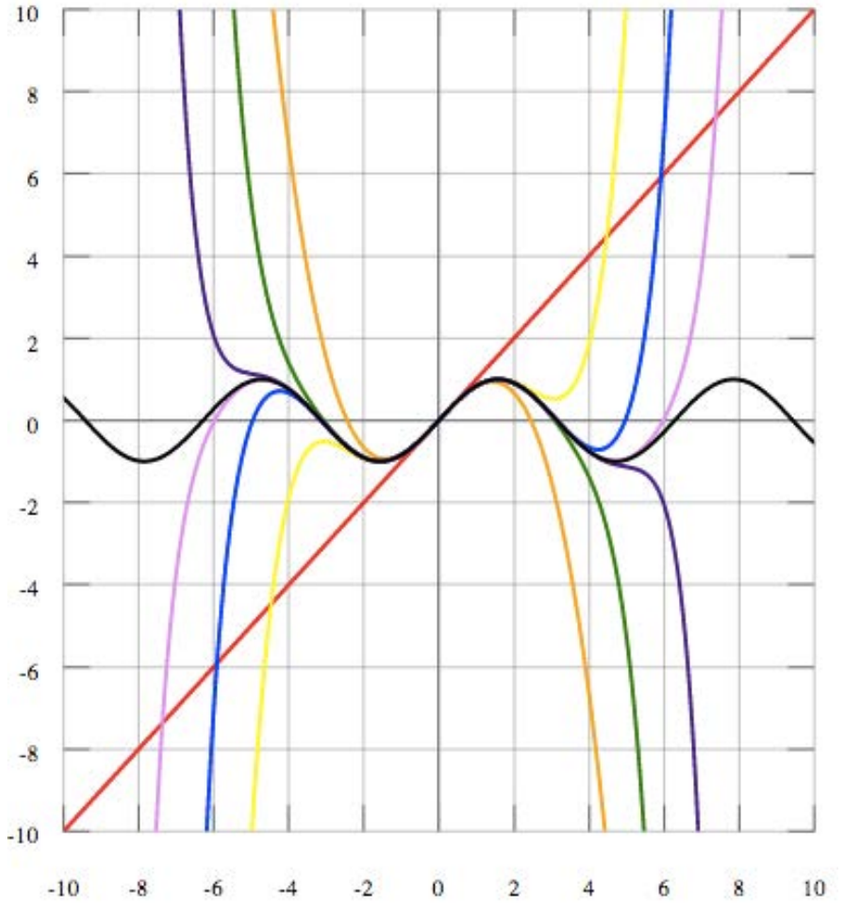

Figure shows sin(x) and Taylor approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13. As more power functions with larger exponents are added, the Taylor polynomial approaches the correct function.

EXAMPLE

Functions of the form $f(x) = x^3, f(x) = x^{1.2}, f(x) = x^{-4}$ are all power functions.

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Taylor and Maclaurin Series

Taylor series represents a function as an infinite sum of terms calculated from the values of the function's derivatives at a single point.

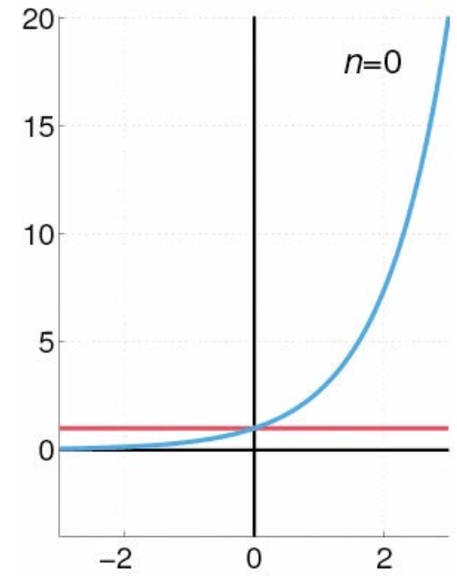
KEY POINTS

- Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial.
- A function that is equal to its Taylor series in an open interval (or a disc in the complex plane) is known as an analytic function.
- The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable in a neighborhood of a real or complex number a is the power series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. If a = 0, the series is called a Maclaurin series.

A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was formally introduced by the English mathematician Brook Taylor in 1715. If the Taylor series is centered at zero, then that series is also called a Maclaurin series, named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

It is common practice to approximate a function by using a finite number of terms of its Taylor series. Taylor's theorem gives quantitative estimates on the error in this approximation. Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial. The Taylor series of a function is the limit of that function's Taylor polynomials, provided that the limit exists. (*Figure 4.27*) A function may not be equal to its Taylor series, even if its Taylor series converges at every point. A function that is equal to its Taylor series in an open interval (or a disc in the complex plane) is known as an **analytic function**.

Figure 4.27 Exponential Function as a Power Series



The exponential function (in blue) and the sum of the first n+1 terms of its Taylor series at 0 (in red) up to n=8.

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely **differentiable** in a neighborhood of a real or complex number a is the power series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where n! denotes the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point x=a. The derivative of order zero f is defined to be f itself, and $(x - a)^0$ and 0! are both defined to be 1. In the case that a = 0, the series is also called a Maclaurin series.

Examples

1. The Maclaurin series for $(1 - x)^{-1}$ for $|x| < 1$ is the geometric series: $1 + x + x^2 + x^3 + \cdots$, so the Taylor series for x^{-1} at a = 1 is $1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots$.

2. The Taylor series for the exponential function e^x at a = 0 is

$$1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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Applications of Taylor Series

Taylor series expansion can help approximating values of functions and evaluating definite integrals.

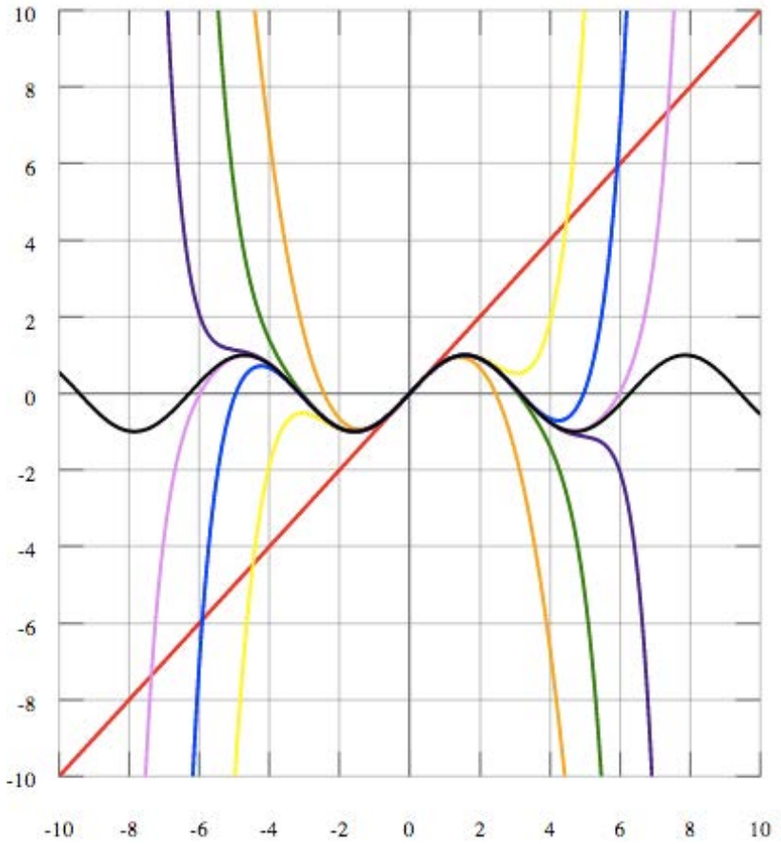
KEY POINTS

- The partial sums of the series, which is called the Taylor polynomials, can be used as approximations of the entire function.
- Differentiation and integration of power series can be performed term by term, and hence could be easier than directly working with the original function.
- The (truncated) series can be used to compute function values numerically. This is particularly useful in evaluating special mathematical functions (such as Bessel function).

Uses of the Taylor series for **analytic functions** include:

1. The partial sums (the Taylor polynomials) of the series can be used as approximations of the entire function. These approximations are often good enough if sufficiently many terms are included. (*Figure 4.28*) Approximations using the first few terms of a Taylor series can make otherwise unsolvable problems possible for a restricted domain; this approach is often used in physics.

Figure 4.28 Taylor Polynomials



As more terms are added to the Taylor polynomial, it approaches the correct function. This image shows sin x and its Taylor approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13.

2. Differentiation and integration of power series can be performed term by term and is hence particularly easy. Taylor series is especially useful in evaluating **definite integrals**. For example, to evaluate $\int_0^1 e^{x^3}\,dx$, Taylor series expansion of $e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$ and the substitution of $t = x^3$ can be used. Since each term in the summation can be integrated separately, we can evaluate the definite integral as long as the sum converges.

3. The (truncated) series can be used to compute function values numerically. This is particularly useful in evaluating special mathematical functions (such as Bessel function).

4. Algebraic operations can be done readily on the power series representation; for instance the Euler's formula follows from Taylor series expansions for trigonometric and exponential functions. From the Taylor expansion of cos(x), sin(x), and exp(ix)

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

we get,

$$cos(x) + i\,sin(x) = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

$$= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$= e^{ix}$. This result is of fundamental importance in many fields of mathematics (for example, in **complex analysis**), physics and engineering.

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Summing an Infinite Series

Infinite sequences and series can either converge or diverge.

KEY POINTS

- Infinite sequences and series continue indefinitely.
- A series is said to converge when the sequence of partial sums has a finite limit.
- A series is said to diverge when the limit is infinite or does not exist.

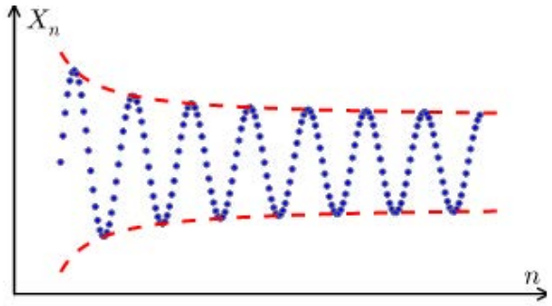
A series is the sum of the terms of a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Unlike finite summations, infinite series need tools from mathematical analysis, and specifically the notion of limits, to be fully understood and manipulated. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, and finance.

For any infinite sequence of real or complex numbers, the associated series is defined as the ordered formal sum: $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$. The sequence of partial sums $\{S_k\}$

associated to a series $\sum_{n=0}^{\infty} a_n$ is defined for each k as the sum of the sequence $\{a_n\}$ from a_0 to a_k: $S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k$.

Infinite sequences and series can either converge or diverge. A series is said to converge when the sequence of partial sums has a finite limit. By definition the series $\sum_{n=0}^{\infty} a_n$ converges to a limit L if and only if the associated sequence of partial sums $\{S_k\}$ converges to L. This definition is usually written as: $L = \sum_{n=0}^{\infty} a_n \Leftrightarrow L = \lim_{k\to\infty} S_k$. If the limit of is infinite or does not exist, the series is said to diverge, see *Figure 4.29*.

Figure 4.29 Infinite sequence



An infinite sequence of real numbers shown in blue dots. This sequence is neither increasing, nor decreasing, nor convergent, nor Cauchy. It is, however, bounded.

An easy way that an infinite series can converge is if all the a_n are zero for sufficiently large n's. Such a series can be identified with a finite sum, so it is only infinite in a trivial sense. Working out the properties of the series that converge even if infinitely many terms are non-zero is, therefore, the essence of the study of series. In the following atoms, we will study how to tell whether a series converges or not and how to compute the sum of a series when such a value exists.

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Convergence of Series with Positive Terms

For a sequence $\{a_n\}$, where a_n is a non-negative real number for every n, the sum $\sum_{n=0}^{\infty} a_n$ can either converge or diverge to ∞.

KEY POINTS

- Because the partial sum S_k of a series with non-negative terms can only increase as k becomes larger, the limit of the partial sum can either converge or diverge to ∞.
- A geometric sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$ converges to 2, which can be understood visually.
- The series $(b_n = \frac{1}{n-1} - \frac{1}{n})$, which is known to converge.

For a sequence $\{a_n\}$, where a_n is a non-negative real number for every n, the sequence of partial sums $S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k$ is non-decreasing. Because the partial sum S_k can only increase as k becomes larger, the limit of the partial sum can either converge or diverge to ∞. Therefore, it follows that a series $\sum_{n=0}^{\infty} a_n$ with non-negative terms converges if and only if the sequence S_k of partial sums is bounded.

Example 1

The series $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent because of the inequality

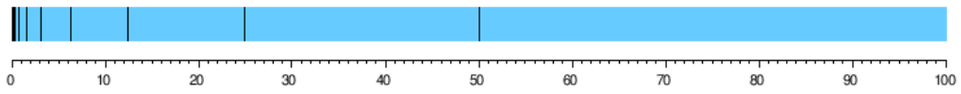
$$\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}, (n \geq 2)$$ and because

$$\sum_{n\geq 2} (\frac{1}{n-1} - \frac{1}{n}) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots = 1.$$

Example 2

Would the series $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots]$ converge? It is possible to "visualize" its convergence on the real number line (*Figure 4.30*)? We can imagine a line of length 2, with successive segments marked off of lengths 1, ½, ¼, etc. There is always room to mark the next segment, because the amount of line remaining is always the same as the last segment marked: when we have marked off ½, we still have a piece of length ½ unmarked, so we can certainly mark the next ¼. This argument does not prove that the sum is equal to 2 (although it is), but it does prove that it is at most

Figure 4.30 Geometric Sum



Visualization of the geometric sum. The length of the line (=2) can contain all the successive segments marked off of lengths 1, ½, ¼, etc.

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2. In other words, the series has an upper bound. Proving that the series is equal to 2 requires only elementary algebra, however. If the series is denoted S, it can be seen that

$S/2 = \frac{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}{2} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$. Therefore,

$S - S/2 = 1 \Rightarrow S = 2$.

For these specific examples, there are easy ways to check the **convergence**. However, it could be the case that there are no easy ways to check the convergence. For these general cases, we can experiment with several well- known **convergence tests** (such as ratio test, integral test, etc.). We will learn some of these tests in the following atoms.

Chapter 5

Advanced Topics in Single-Variable Calculus and an Introduction to Multivariable Calculus

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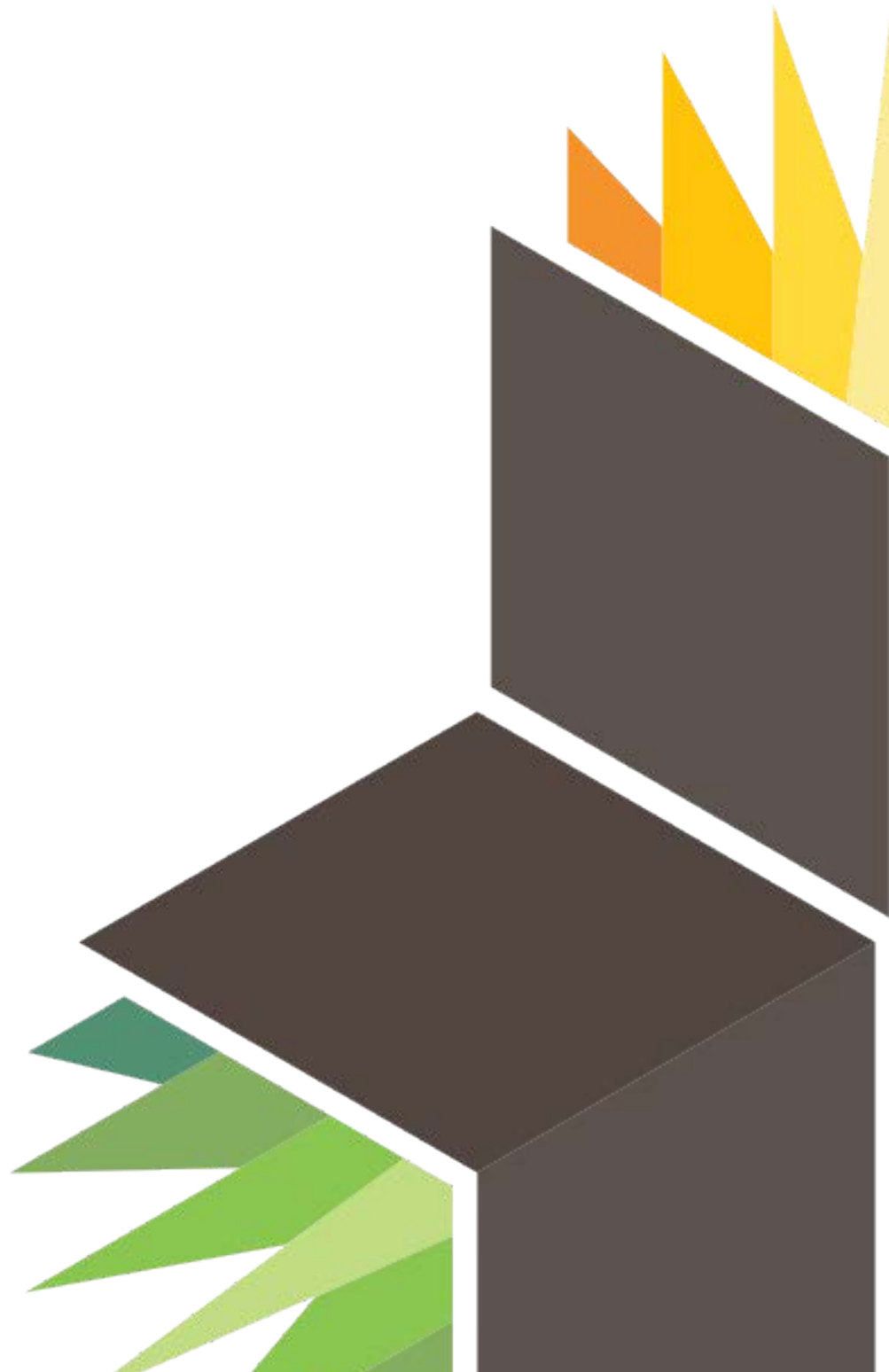

Section 1

Vectors and the Geometry of Space

Three-Dimensional Coordinate Systems

Vectors in the Plane

Vectors in Three Dimensions

The Dot Product

The Cross Product

Equations of Lines and Planes

Cylinders and Quadric Surfaces

Cylindrical and Spherical Coordinates

Surfaces in Space

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Three-Dimensional Coordinate Systems

The three dimensional coordinate system expresses a point in space with three parameters, often length, width and depth (x,y, and z).

KEY POINTS

- There are many types of coordinate systems, including Cartesian, spherical, and cylindrical coordinates.
- In the Cartesian system, all three of the parameters are represented as the quantitative distance from the reference plane.
- In order to convert from Cartesian to spherical, you need to convert each parameter separately, as follows:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos(\frac{z}{r})$$

$$\varphi = \arctan(\frac{y}{x})$$

The Three Dimensional Coordinate System

A three dimensional space has three geometric parameters: x, y, and z. These are often referred to as length, width and depth. Each parameter is perpendicular to the other two, and cannot lie in the same plane. *Figure 5.1* shows a Cartesian **coordinate system** that uses the parameters x, y, and z.

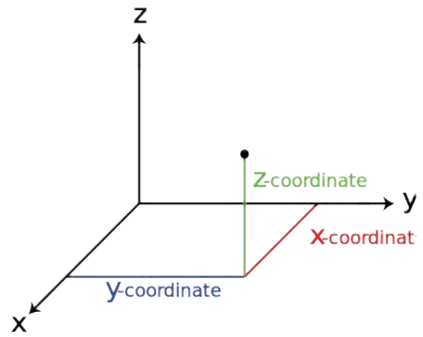


Figure 5.1 Three-Dimensional Space
This is a three dimensional space represented by a Cartesian coordinate system.

Cartesian Geometry

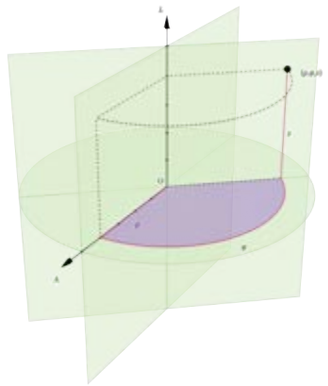
Also known as analytical geometry, this system is used to describe every point in three dimensional space in three parameters, each perpendicular to the other two at the **origin**. Each parameter is labeled relative to its axis with a quantitative representation of its distance from its plane of reference, which is determined by the other two parameter axes.

Other Coordinate Systems

- Cylindrical Coordinates: [ρ,φ,z] (*Figure 5.2*) The cylindrical system uses two linear parameters and one radial parameter: ρ - the radial distance from the point to z; φ - the angle between the reference direction and the point; and z - the distance from the reference plane to the point.
- Spherical Coordinates: [r,θ,φ] (*Figure 5.3*) The spherical system is used commonly in mathematics and physics, but we

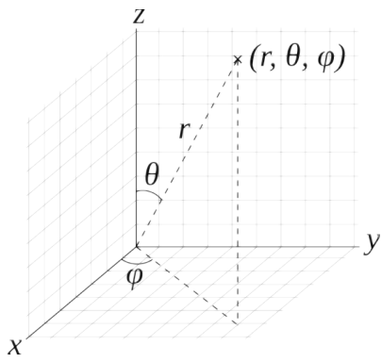
will cover it further in another atom: r - the radial distance from the origin to the point; θ - the angle between the zenith direction and directional vector of r; and φ - the angle from the reference direction to the orthogonal plane projected by the directional vector of r.

Figure 5.2 Cylindrical Coordinate System



The cylindrical coordinate system is like a mix between the spherical and Cartesian system, incorporating linear and radial parameters.

Figure 5.3 Spherical Coordinate System



The spherical system is used commonly in mathematics and physics and has variables of r, θ, and ϕ.

Cartesian to Spherical

Often, you will need to be able to convert from spherical to cartesian, or the other way around. The following equations will allow you to do just that:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos(\frac{z}{r})$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

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Vectors in the Plane

Vectors are needed in order to describe a plane and can give the direction of all dimensions in one vector equation.

KEY POINTS

- In order to adequately describe a plane, you need more than a point--you need its normal vector.
- The normal vector is perpendicular to the directional vector of the reference point.
- You can find the equation of a vector that describes a plane by using the following equation: a (x-xo) + b(y - yo) +c(z - zo) = 0.

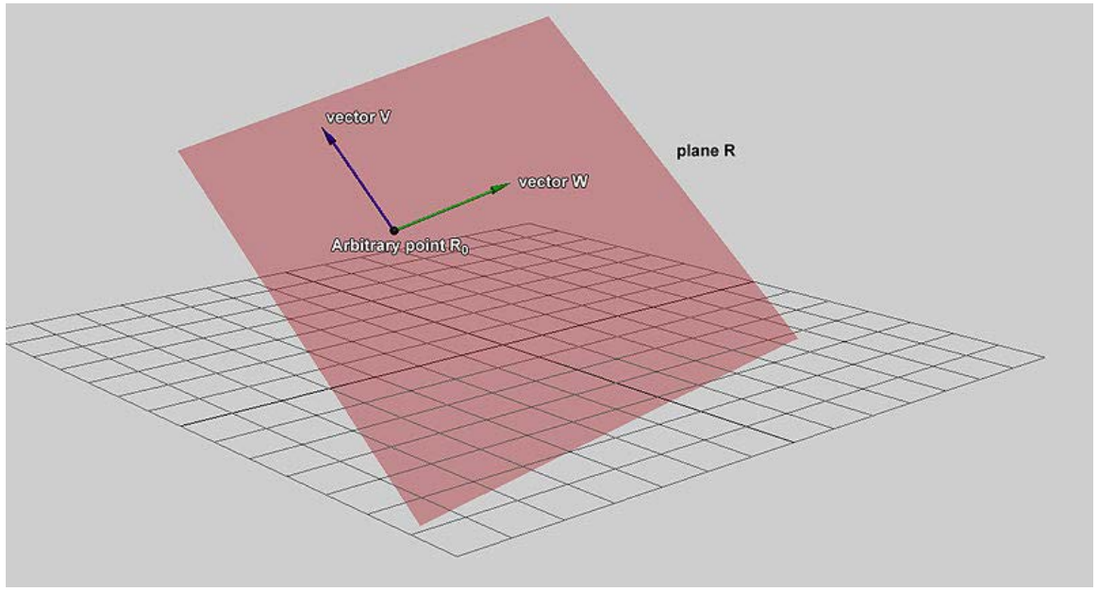
Vectors are used to describe the direction of lines in one or more dimensions, usually two or three. *Figure 5.4* shows a plane and the vector that describes it. This is the **normal** vector, which was covered in more depth in a separate atom. This normal vector occurs at point Po on a plane at an arbitrary point, P. In order to obtain the equation of the normal vector of a specific plane, you can use the following equations:

$n \bullet \vec{r} = n \bullet \vec{r_o}$, where r is the position factors to P and Po and can be used as the scalar:

$$n = < a, b, c > r = < x, y, z > ro = < x_0, y_0, z_0 > < a, b, c > \bullet$$

$$< x - x_0, y - y_0, z - z_0 > = 0a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Figure 5.4 Normal Vector to a Plane



In order to describe a plane, you need the normal vector to that plane.

Example

Find the equation of the plane that passes through Q = (5,2,8,); R = (6,-2,9); P = (0,7,3).

$\vec{a} = \overrightarrow{QR} = (1,0,1)$

$\vec{b} = \overrightarrow{QP} = (-5,5,-11)$

n = a x b = (0 - 5), (-5 + 11), (5 - 0) = -5i + 6j + 5k = -5 (x- 5) + 6 (y - 2) + 5 (z - 8) = -5x + 25 + 6y - 12 + 5z - 40 = -5x + 6y + 5z = 27

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Vectors in Three Dimensions

A Euclidean vector is a geometric object that has magnitude (or length) and direction.

KEY POINTS

- Vectors play an important role in physics.
- In the Cartesian coordinate system, a vector can be represented by identifying the coordinates of its initial and terminal point.
- The coordinate representation of vectors allows the algebraic features of vectors to be expressed in a convenient numerical fashion.
- Vectors can be added to other vectors according to vector algebra.

A Euclidean vector (sometimes called a geometric or spatial vector, or—as here—simply a vector) is a geometric object that has magnitude (or length) and direction and can be added to other vectors according to vector algebra. A Euclidean vector is frequently represented by a line segment with a definite direction, or graphically as an arrow, connecting an initial point A with a terminal point B, and denoted by $\overrightarrow{AB}$.

Vectors play an important role in physics: velocity and acceleration of a moving object and forces acting on it are all described by vectors. Many other physical quantities can be usefully thought of as vectors. The mathematical representation of a physical vector depends on the coordinate system used to describe it. Other vector-like objects that describe physical quantities and transform in a similar way under changes of the coordinate system include pseudovectors and tensors.

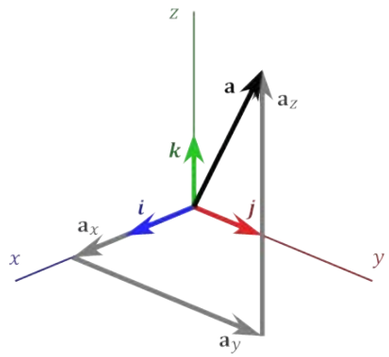
In the Cartesian coordinate system, a vector can be represented by identifying the coordinates of its initial and terminal point (*Figure 5.5*). For instance, in three dimensions, the points A = (1,0,0) and B = (0,1,0) in space determine the free vector $\overrightarrow{AB}$ pointing from the point x=1 on the x-axis to the point y=1 on the y-axis. Typically in Cartesian coordinates, one considers primarily bound vectors. A bound vector is determined by the coordinates of the terminal point, its initial point always having the coordinates of the origin O = (0,0,0). Thus the bound vector represented by (1,0,0) is a vector of unit length pointing from the origin along the positive x-axis. The coordinate representation of vectors allows the algebraic features of vectors to be expressed in a convenient numerical fashion. For example, the sum of the vectors (1,2,3) and (−2,0,4) is the vector

$$(1,2,3) + (-2,0,4)$$

$$= (1 - 2,2 + 0,3 + 4)$$

$$= (-1,2,7).$$

Figure 5.5 Vector in 3D Space



A vector in the 3D Cartesian space, showing the position of a point A represented by a black arrow. i,j,k are unit vectors in x,y,z axis, respectively

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The Dot Product

The dot product takes two vectors of the same dimension and returns a single value.

KEY POINTS

- The dot product can be found algebraically or geometrically. The algebraic method employs the sum of the products of corresponding parameters, and the geometric method uses the product of the magnitudes of the vectors and the cosine of the angle between them.
- The dot product is a commutative property.
- The dot product is a distributive property.

The dot product takes two vectors and returns a single value. The dot product can only be taken from two vectors of the same dimension.

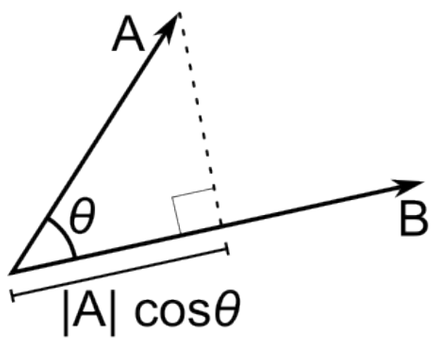

Figure 5.6 Dot Product

When finding the dot product geometrically, you need the magnitudes of the vectors and the angle between them.

The dot product is the sum of the product of the corresponding parameters. Geometrically, the dot product is the product of the magnitudes of two vectors and the cosine of the angle between them. *Figure 5.6* shows all of the components that are needed to take the dot product of two vectors. This is different from the **cross product**, which gives an answer in vector form.

There are two representations of the dot product:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \vec{A} \cdot \vec{B} = |A||B| \cos\theta$$

Properties

The dot product is a **commutative** property, which means that the order of the terms does not change the outcome.

- The dot product is a commutative property, which means that the order of the terms does not change the outcome.

  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

- The dot product is a distributive property.

  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

- If two vectors are normal (perpendicular) to each other, their dot product will be equal to zero. $\vec{a} \cdot \vec{b} = 0$

Example

Find the dot product of the two vectors Q (5,2,8) and R (6,-2,9)

$$\vec{Q} \cdot \vec{R} = Q_1 R_1 + Q_2 R_2 + Q_3 R_3$$

$$= 5 \cdot 6 + 2 \cdot (-2) + 8 \cdot 9$$

$$= 98$$

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The Cross Product

The cross product is an operation whose answer is in vector form and represents the magnitude and direction of the product of the vectors.

KEY POINTS

- The right hand rule can be used to find the direction of vector c, the cross product of vectors a and b.
- The area of a parallelogram made by two vectors is the magnitude of vector c.
- The cross product can be found by using one of the following two equations:
  $a \ x \ b = |a||b|\sin\theta$
  $= < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 > .$

The cross product is a binary operation of two three-dimensional vectors. The result is a vector which is **perpendicular** to both of the original vectors. Because it is perpendicular to both original vectors, the resulting vector is normal to the plane of the original vectors.

If the two original vectors are parallel to each other, the cross product will be zero.

The cross product is denoted as a x b = c.

The direction of vector c can be found by using the right hand rule, which is demonstrated in *Figure 5.7*. The magnitude of vector c is equal to the area of the **parallelogram** made by the two original vectors.

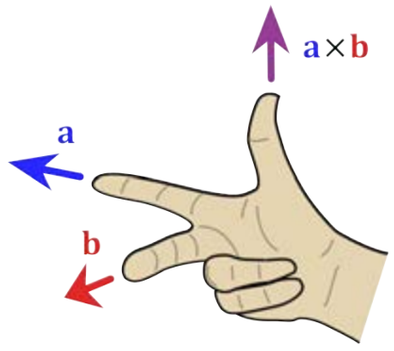

Figure 5.7 The Right Hand Rule
If you use the rules shown in the figure, your thumb will be pointing in the direction of vector c, the cross product of the vectors.

The cross product is different from the dot product because the answer is in vector form in the same number of dimensions as the original two vectors, where the dot product is given in the form of a single quantity in one dimension.

The cross product can be found both algebraically and geometrically.

Geometric Method

The geometric method of finding the cross product uses the magnitudes of the vectors and the sine of the angle between them.

$$a \ x \ b = |a||b|\sin\theta$$

Algebraic Method

The algebraic method of finding the cross product of two vectors involves inputting the vector information into matrices and manipulating them:

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} k = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

The manipulated matrices form the following equations:

$$= < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

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Equations of Lines and Planes

A line is a vector which connects two points on a plane and the direction and magnitude of a line determine the plane on which it lies.

KEY POINTS

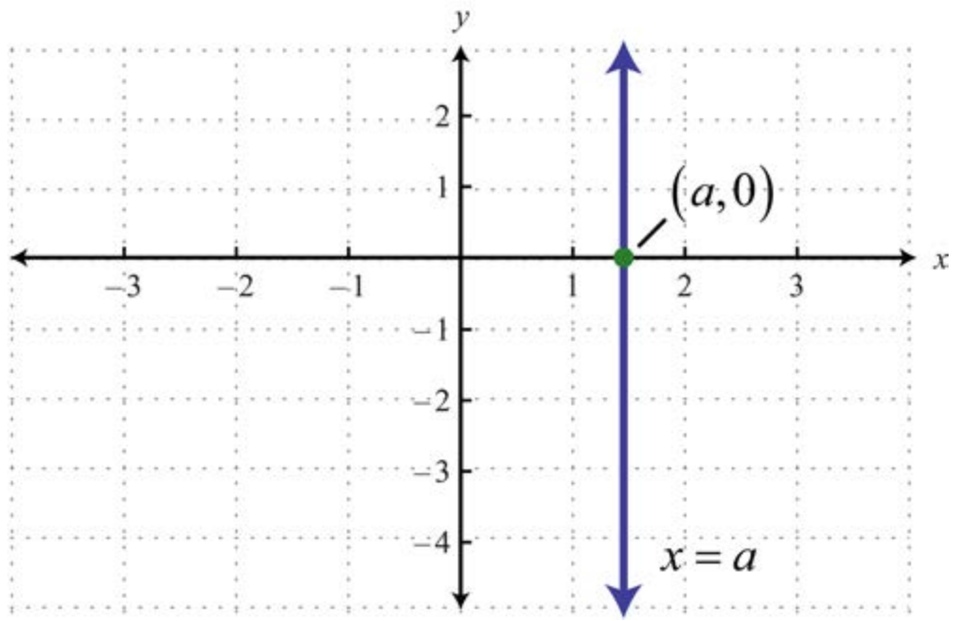
- The slope of the line, and the plane it lies on, is the angle of inclination of that line.
- A line is a two dimensional representation of a three dimensional geometric object, a plane.
- The parametric equation of a line can be found using the following equation:

$$< x, y, z > = < x_0 + ta, y_0 + tb, z_0 + tc > x = x_0 + at; y = y_0 + bt; z = z_0 + ct$$

Lines and Planes

A line is described by a point on the line and its angle of inclination, or slope. Every line lies in a plane which is determined by both the direction and slope of the line. A line is essentially a representation of a cross section of a plane, or a two dimensional version of a plane which is a three dimensional object.

Figure 5.8 Vertical Line, Graphed



Vertical line x = a, lying on the x-y plane (z=0).

Equations of Lines and Planes

The components of equations of lines and planes are as follows:

A line in three dimensional space is given by a point, $P_0(x_0, y_0, z_0)$, or a plane, M and the direction of the plane M. This direction is described by a vector, v, which is parallel to plane M and P is the arbitrary point on plane M.

The position vector of point P_0 is called ro and the position vector of point P is called r. The vector from P to P_0 is called vector a.

Vectors a and v are parallel to each other.

Now, we can use all this information to form the equation on a line on plane M.

The vector equation of a line is:

$r = r_0 + tv$

Where:

t - represents the location of vector r on plane M.

Parametric equations

$< x, y, z > = < x_0 + ta, y_0 + tb, z_0 + tc > x = x_0 + at; y = y_0 + bt; z = z_0 + ct$

Figure 5.8 shows a line lying in the x-y plane.

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Cylinders and Quadric Surfaces

A quadric surface is any D-dimensional hypersurface in (D + 1)-dimensional space defined as the locus of zeros of a quadratic polynomial.

KEY POINTS

- A cylinder is one of the most basic curvilinear geometric shapes, the surface formed by the points at a fixed distance from a given line segment, the axis of the cylinder.
- A cylinder can be seen as a polyhedral limiting case of an n-gonal prism where n approaches infinity.
- A quadric, or quadric surface, is any D-dimensional hypersurface in (D + 1)-dimensional space defined as the locus of zeros of a quadratic polynomial.
- Cylinder, sphere, ellipsoids, etc. are special cases of quadric surfaces.

Cylinder

A cylinder (from Greek "roller" or "tumbler") is one of the most basic curvilinear geometric shapes. The surface is formed by the points at a fixed distance from a given line segment, the axis of the cylinder (*Figure 5.9*). The solid enclosed by this surface and by two

planes perpendicular to the axis is also called a cylinder. The surface area and the volume of a cylinder have been known since antiquity. A cylinder can be seen as a polyhedral limiting case of an n-gonal prism where n approaches infinity.

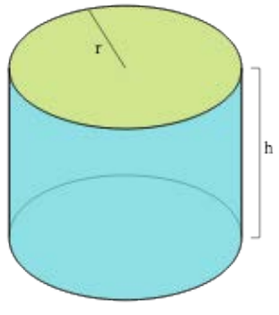


Figure 5.9 Cylinder
A right circular cylinder with radius r and height h.

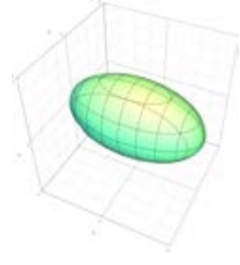
In common use, a cylinder is taken to mean a finite section of a right circular cylinder, i.e. the cylinder with the generating lines perpendicular to the bases, with its ends closed to form two circular surfaces, as in the figure right. If the cylinder has a radius r and length (height) h, then its volume is given by $V = \pi r^2 h$, and its surface area is $A = 2\pi rh$ without the top and bottom, and $2\pi r(r + h)$ with them.

Quadric Surface

A quadric, or quadric surface, is any D-dimensional **hypersurface** in (D + 1)-dimensional space defined as the locus of zeros of a quadratic polynomial. In coordinates $\{x_1, x_2,..., x_{D+1}\}$, the general quadric is defined by the algebraic equation

$$\sum_{i,j=1}^{D+1} x_i Q_{ij} x_j + \sum_{i=1}^{D+1} P_i x_i + R = 0.$$ Cylinder, sphere, ellipsoids, etc. are special cases of quadric surfaces.

Figure 5.10 Ellipsoid



An ellipsoid given as $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Examples

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (*Figure 5.10*)
- Sphere: $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$
- Elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$
- Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
- Parabolic cylinder: $x^2 + 2ay = 0$

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Cylindrical and Spherical Coordinates

Cylindrical and spherical coordinates are useful when describing objects or phenomena with specific symmetries.

KEY POINTS

- A cylindrical coordinate system is a three-dimensional coordinate system that specifies point positions by three numbers ρ, φ, z.
- Cylindrical coordinates are useful in connection with objects and phenomena that have some rotational symmetry about the longitudinal axis.
- A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers r, θ, φ.

While **Cartesian** coordinates have many applications, cylindrical and spherical coordinates are useful when describing objects or phenomena with specific symmetries.

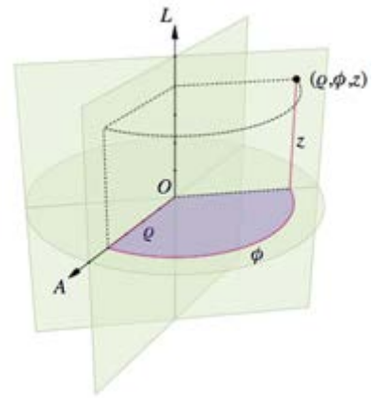
Cylindrical Coordinates

A cylindrical coordinate system is a three-dimensional coordinate system that specifies point positions by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis (*Figure 5.11*). The latter distance is given as a positive or negative number, depending on which side of the reference plane faces the point.

Cylindrical coordinates are useful in connection with objects and phenomena that have some rotational symmetry about the longitudinal axis, such as water flow in a straight pipe with a round cross-section, heat distribution in a metal cylinder, electromagnetic fields produced by an electric current in a long, straight wire, and so on.

For the conversion between cylindrical and Cartesian coordinate co-ordinates, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation z = 0), and the

Figure 5.11 Cylindrical Coordinate System



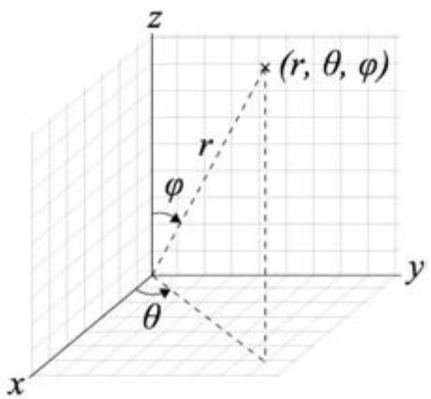
A cylindrical coordinate system with origin O, polar axis A, and longitudinal axis L. The dot is the point with radial distance ρ = 4, angular coordinate ϕ = 130°, and height z = 4.

cylindrical axis is the Cartesian z axis. Then the z coordinate is the same in both systems, and the correspondence between cylindrical (ρ,φ) and Cartesian (x,y) are the same as for polar coordinates, namely $x = \rho \cos \varphi y = \rho \sin \varphi$. In the reverse transformation,

$$\rho = \sqrt{x^2 + y^2}$$

$$\varphi = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ \arcsin(\frac{y}{\rho}) & \text{if } x \geq 0 \\ -\arcsin(\frac{y}{\rho}) + \pi & \text{if } x < 0 \end{cases}$$

Figure 5.12 Spherical Coordinate System



Spherical coordinates (r, θ, ϕ) as often used in mathematics: radial distance r, azimuthal angle θ, and polar angle ϕ. The meanings of θ and ϕ have been swapped compared to the physics convention.

Spherical Coordinates

A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the **azimuth** angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane (*Figure 5.12*). Spherical coordinates are useful in connection with objects and phenomena that have spherical symmetry, such as an electric charge located at the origin.

The spherical coordinates (radius r, inclination θ, azimuth φ) of a point can be obtained from its Cartesian coordinates (x, y, z) by the formulae:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos\left(\frac{z}{r}\right)$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

In reverse,

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

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Surfaces in Space

A surface is a two-dimensional, topological manifold.

KEY POINTS

- To say that a surface is "two-dimensional" means that, about each point, there is a coordinate patch on which a two-dimensional coordinate system is defined.
- The concept of surface finds application in physics, engineering, computer graphics, and many other disciplines, primarily in representing the surfaces of physical objects.
- Surfaces could be the locus of zeros of certain functions, usually polynomial functions.

A surface is a two-dimensional, topological **manifold**. The most familiar examples are those that arise as the boundaries of solid objects in ordinary three-dimensional Euclidean space R^3 — for example, the surface of a ball. On the other hand, there are surfaces, such as the Klein bottle, that cannot be embedded in three-dimensional Euclidean space without introducing singularities or self-intersections.

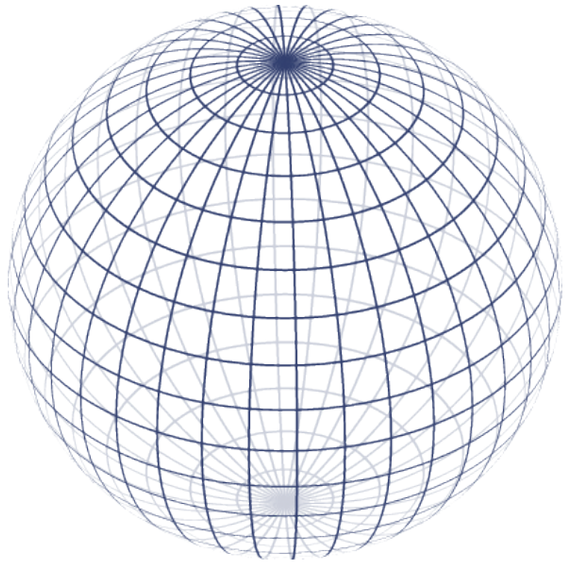
To say that a surface is "two-dimensional" means that, about each point, there is a coordinate patch on which a two-dimensional coordinate system is defined. For example, the surface of the Earth

is (ideally) a two-dimensional surface, and latitude and longitude provide two-dimensional coordinates on it (except at the poles and along the 180th **meridian**).

The concept of surface finds application in physics, engineering, computer graphics, and many other disciplines, primarily in representing the surfaces of physical objects. For example, in analyzing the aerodynamic properties of an airplane, the central consideration is the flow of air along its surface.

Historically, surfaces were initially defined as subspaces of Euclidean spaces. Often, these surfaces were the locus of zeros of certain functions, usually polynomial functions. Such a definition considered the surface as part of a larger (Euclidean) space, and as such was termed extrinsic.

Figure 5.13 A Sphere Defined Parametrically



A sphere can be defined parametrically (by $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$) or implicitly (by $x^2 + y^2 + z^2 - r^2 = 0$.)

EXAMPLE

In the Cartesian coordinates, a sphere can be defined parametrically (by $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$) or implicitly by $x^2 + y^2 + z^2 - r^2 = 0$ (*Figure 5.13*). In spherical coordinates, the surface can be expressed simply by $r=R$.

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Section 2

Vector Functions

Vector-Valued Functions

Arc Length and Speed

Calculus of Vector-Valued Functions

Arc Length and Curvature

Planetary Motion According to Kepler and Newton

Tangent Vectors and Normal Vectors

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Vector-Valued Functions

A vector function covers a set of multidimensional vectors at the intersection of the domains of f, g and h.

KEY POINTS

- A vector valued function can be made up of vectors and/or scalars.
- Each component function in a vector valued function represents the location of the value in a different dimension.
- The domain of the vector value function is the intersection of the component function domains.
- Vector valued functions can behave the same ways as vectors, and be evaluated similarly.

What is a Vector Valued Function?

Also called **vector** functions, vector valued functions allow you to express the position of a point in multiple dimensions within a single function. These can be expressed in an infinite number of dimensions, but are most often expressed in two or three. The input into a vector valued function can be a vector or a **scalar**. In this atom we are going to introduce the properties and uses of the vector valued functions.

Properties of Vector Valued Functions

A vector valued function allows you to represent the position of a particle in one or more dimensions. A three- dimensional vector valued function requires three functions, one for each dimension. In Cartesian form with standard unit vectors (i,j,k), a vector valued function can be represented in either of the following ways:

$$r(t) = f(t)i + g(t)j + h(t)kr(t) = < f(t), g(t), h(t) >$$

where t is being used as the variable (*Figure 5.14*). This is a three dimensional vector valued function. The graph shows a visual representation of

$$r(t) = < 2cos(t), 4sin(t), t >$$

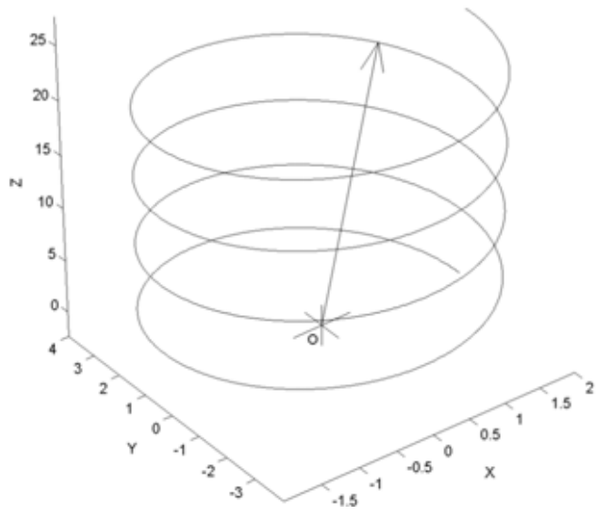

Figure 5.14
Example of Vector-Valued Function
This is the graph of the vector valued function r(t)=<2cost,4sint,t>. This can also be written as x(t)=2cost, y(t)=4sint, z(t)=t

This can be broken down into three separate functions called component functions:

$$x(t) = 2cos(t) y(t) = 4sin(t) z(t) = t$$

If you were to take a cross section of *Figure 5.14*, with the cut perpendicular to any of the three axes, you would see the graph of that function. For example, if you were to slice the three-dimensional shape perpendicular to the z-axis, the graph you would see would be of the function z(t)=t.The **domain** of a vector valued function is a domain that satisfies all of the component functions. It can be found by taking the intersection of the individual component function domains. The vector valued functions can be manipulated in the same way as a vector; they can be added, subtracted, and the dot product and the cross product can be found.

EXAMPLE

For this example, we will use time as our parameter. The following vector valued function represents time, t: r(t) = f(t)i + g(t)j + h(t)k. This function is representing a position. Therefore, if we take the derivative of this function, we will get the velocity of this function.

$$\frac{dr(t)}{dt} = f(t)i' + g(t)j' + h(t)k'$$

$$\frac{dr(t)}{dt} = v(t)$$

If we differentiate a second time, we will be left with the following acceleration: $\frac{dv(t)}{dt} = a(t)$

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Arc Length and Speed

Arc length and speed are, respectively, a function of position and its derivative with respect to time.

KEY POINTS

- Arc length is found by placing a number of points along a curve, connecting them by line segments, and then adding those segment lengths together.
- The beginning of deriving the formula for arc length starts with Pythagorus's theorem.
- When finding the arc length, the integral used needs to be with respect to position, x.
- When finding the speed along a curve, the integral used needs to be with respect to time, t.

Arc Length and Speed

Since length is a magnitude that involves position, it is easy to deduce that the **derivative** of a length, or position, will give you the **velocity**-- also known as speed-- of a function. This is because a derivative gives you a rate of change with respect to a parameter. Velocity is the rate of change of a position with respect to time. Let's start this atom by looking at arc length with calculus.

Figure 5.15 Arc Length

The arc length is the equivalent of taking a curve, straightening it out, and then measuring it, as seen in this animation.

Arc Length

The arc length is the length you would get if you took a curve, straightened it out, and then measured the length of that line (*Figure 5.15*). The arc length can be found using geometry, but for the sake of this atom, we are going to use integration. The arc length is approximated by connecting a finite number of points along and curve, connecting those lines to create a a string of very small straight lines, and adding them together. To find this using integration, we should start out by using the Pythagorian Theorem for length of the different sides of a triangle, as seen in *Figure 5.16*:

$$ds^2 = dx^2 + dy^2 \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2} ds = \sqrt{1 + \frac{dy^2}{dx^2}}^2 * dx \rightarrow = \int_a^b \sqrt{1 + f'(x)^2} * dx$$

Where s is the arc length. If x=X(t) and y=Y(t),

$$\int_a^b \sqrt{1 + [f'(x)}^2 * dx \rightarrow s = \int_a^b \sqrt{[X'(t)]^2 + [Y'(t)] * dt} \rightarrow s = lim \sum \sqrt{\triangle x^2 + \triangle y^2} = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} * dt$$

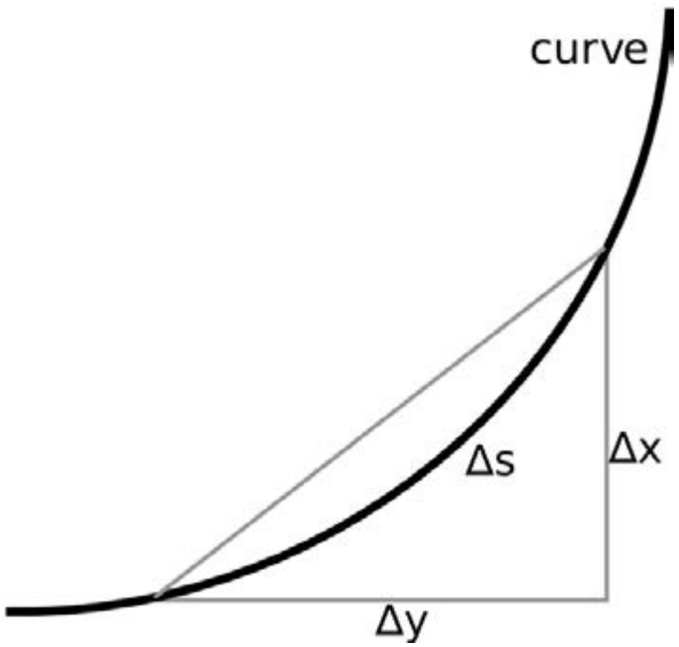

Figure 5.16 Curves and the Pythagorean Theorem

For a small piece of curve, Δs can be approximated with the Pythagorean theorem.

Since this is a function of position and is defined by x, we need to have a derivative that is in respect to x: $\int_a^b \sqrt{1 + \frac{dy}{dx}^2} * dx$

Obviously some cases require polar coordinates instead of Cartesian, and in polar coordinates,

$r = f(\theta)$

Arc Speed

Now that the hard part is over, we can easily find the speed along this curve. Since speed is in relation to time and not position, we need to revert back to the arc length with respect to time:

$$\int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} * dt$$

Then, differentiate with respect to time:

$$v(t) = s' = \sqrt{[X'(t)]^2 + [Y'(t)]^2}$$

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Calculus of Vector-Valued Functions

A vector function is a function that can behave as a group of individual vectors and can perform differential and integral operations.

KEY POINTS

- A vector valued function can be made up of vectors and/or scalars.
- Each component function in a vector valued function represents the location of the value in a different dimension.
- Vector valued functions can behave the same ways as vectors, and be evaluated similarly.
- Vector functions are widely used in the study of electromagnetic fields, gravitation fields, and fluid flow.

Vector Valued Functions

In a previous atom, we covered the definition and components of a **vector** valued function. Before we begin this atom, we will lightly review that atom. Vector valued functions--or simply vector functions-- allow you to express the position of a point in multiple dimensions with a single function. In this atom, we will use three dimensions: x, y and z. The vector functions can be made up of both vector and **scalar** functions. These vector functions can be expressed in the following ways:

$r(t) = f(t)i + g(t)j + h(t)k$

$r(t) = < f(t), g(t), h(t) >.$

Figure 5.17 shows the visual representation of the vector function $r(t) = < 2\cos(t), 4\sin(t), t >$. This can be broken down into three separate functions called component functions: $x(t) = 2\cos(t) y(t) = 4\sin(t) z(t) = t$

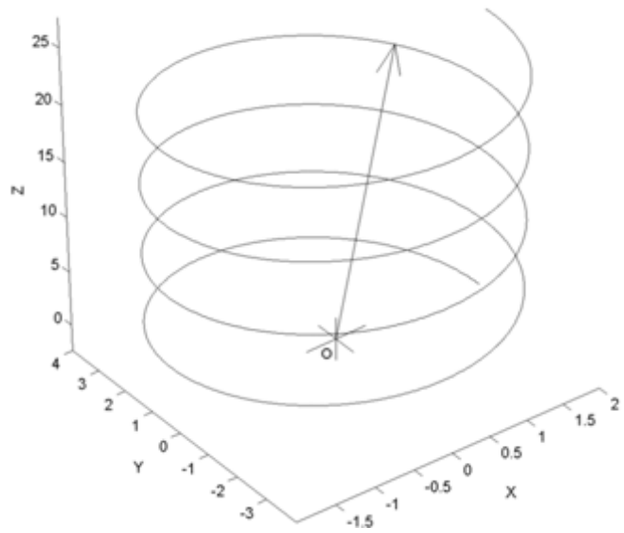

Figure 5.17 Vector Valued Function
This graph is a visual representation of a three dimensional vector valued function.

Vector Calculus

Vector calculus is a branch of mathematics that covers differentiation and integration of vector fields in any number of dimensions. For the purpose of simplicity in this atom, we will only

use three-dimensional vector functions. Because vector functions behave like individual vectors, you can manipulate them the same way you can a vector. Vector calculus is used extensively through out physics and engineering, mostly with regard to electromagnetic fields, gravitational fields, and fluid flow. When taking the derivative of a vector function, the function should be treated as a group of individual functions. For example, let's set up the derivative of vector function r(t):r(t) = f(t)i + g(t)j + h(t)kr'(t) = f'(t)i + g'(t)j + h'(t)k. The integral of a vector function, r(t), can be found by r(t) = f(t)i + g(t)j + h(t)k

$$\int r(t) = \int f(t)dt\,\vec{i} + \int g(t)dt\,\vec{j} + \int h(t)dt\,\vec{k} + \vec{c}$$

Vector functions are used in a number of differential operations, such as:

- Gradients: $\rightarrow grad(f) = \nabla f$. The gradient measures the rate and direction of change in a scalar field.
- Curl: $\rightarrow curl(F) = \nabla X f$. The curl measures the tendency of the vector function to rotate about a point in a vector field.
- Divergence: $\rightarrow div(f) = \nabla \bullet f$. The divergence measures the magnitude of a source at a given point in a vector field.

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Arc Length and Curvature

The curvature of an object is the degree to which it deviates from being flat and can be found using arc length.

KEY POINTS

- The arc length is a function of position, so its derivative will be a function of time. This can give you the rate of change of the position, in relation to time, which is called the curvature.
- The curvature can be found by taking the derivative of thevelocity vector, which is given: $\|\frac{dT}{ds}\|$.
- This same magnitude can also be found using the concept of calculus, the limit.

Arc Curvature

The **curvature** of an arc is a value that represents the direction and **sharpness** of a curve (*Figure 5.18*). On any curve, there is a center of curvature, C. This is the intersection point of two infinitely close **normals** to this curve. The radius, R, is the distance from this intersection point to the center of curvature.

In order to find the value of the curvature, we need to take the parameter time, s, and the unit tangent vector, which in this case is the same as the unit velocity vector, T, which is also a function of time.The curvature is a magnitude of the rate of change of the tangent vector, T:

$$\kappa = \|\frac{dT}{ds}\|$$

Where κ is the curvature and $\frac{dT}{ds}$ is the acceleration vector (the rate of change of the velocity vector over time).

How Does This Relate to Arc Length?

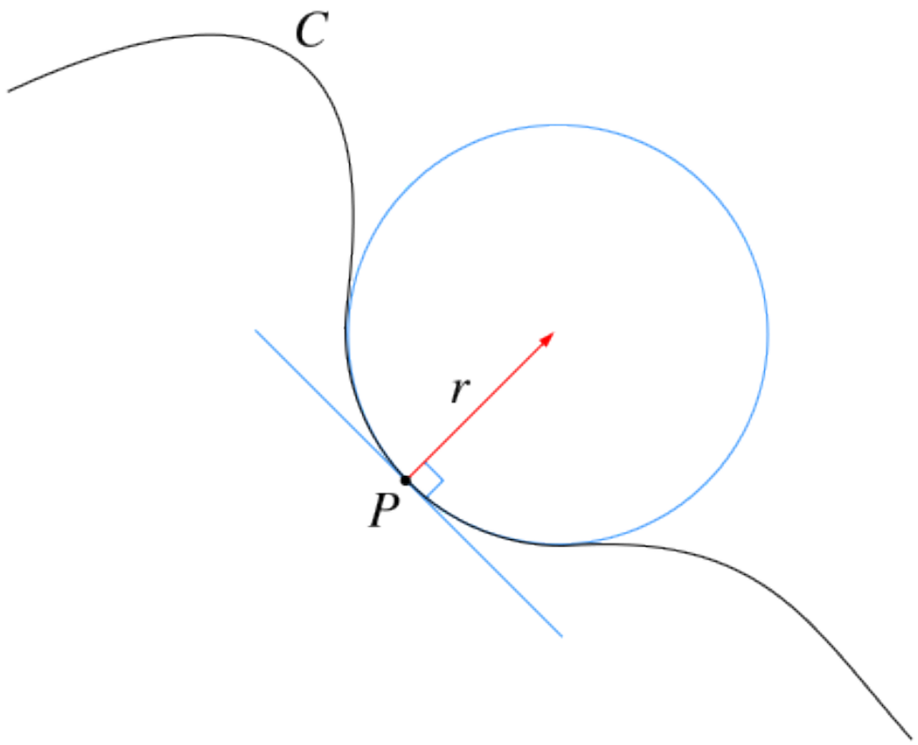
The curvature can also be approximated using limits. Given the points P and Q on the curve, lets call the arc length s(P,Q), and the linear distance from P to Q will be denoted as d(P,Q). The curvature of the arc at point P can be found by obtaining the limit:

$$\kappa(P) = \frac{\lim}{Q \to P} \sqrt{\frac{24 * (s(P,Q) - d(P,Q))}{s(P,Q)^3}}$$

In order to use this formula, you must first obtain the arc length of the curve from points P to Q and length of the linear segment that connect points P and Q. In a previous atom, we went into more detail on how to find the arc length, but for the sake of this atom we will just restate those formulas:

In Cartesian coordinates: $\int_a^b \sqrt{1 + \frac{dy}{dx}^2} * dx$

Figure 5.18 Curvature



Curvature is the amount an object deviates from being flat. Given any curve C and a point P on it, there is a unique circle or line which most closely approximates the curve near P. The curvature of C at P is then defined to be the curvature of that circle or line. The radius of curvature is defined as the reciprocal of the curvature.

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Planetary Motion According to Kepler and Newton

Kepler explained that the planets move in an ellipse around the Sun, which is at one of the two foci of the ellipse.

KEY POINTS

- Kepler's first law of planetary motion describes the motion of its orbit around the Sun.
- Kepler's second law of planetary motion explains the reason why the planet moves faster as it approaches the Sun, and slower as it moves farther away.
- Kepler's third law of planetary motion explains how the period of an orbit is related to the semi-major axis of its orbit.
- Newton takes the information presented by Kepler and uses it to explain that the value of a force on an object is the product of its mass and its orbital acceleration.

Johannes Kepler describes planetary motion with three laws:

1. The orbit of every planet is in an elliptical shape, with the Sun being at one of the two foci of this **ellipse**, called the occupied focus.

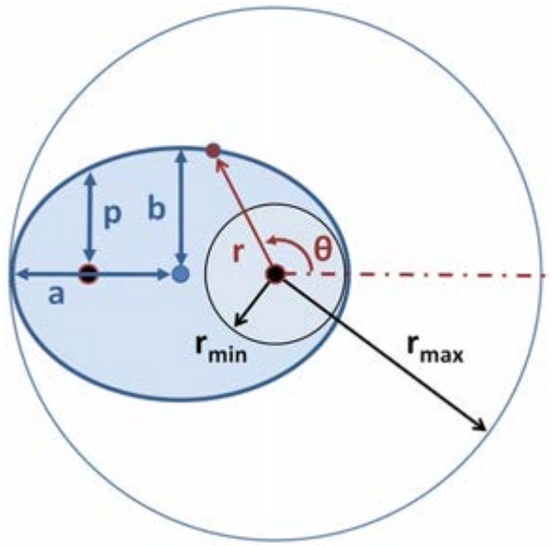
2. If a line were to be drawn from the planet to the Sun, that line would sweep out an equal amount of area during equal intervals of time.

3. The square of the orbital period is proportional to the cube of the semi-major axis of the planet's orbit.

Kepler's First Law

As we already stated, the first law of planetary motion states that the orbit of every planet is an ellipse with the Sun at one focus. In order to discuss this law, and the laws that follow, we should examine the components of an ellipse a bit more closely. Look at *Figure 5.19*. The **eccentricity** of an ellipse tells you how stretched out the ellipse is. The eccentricity can be from 0 to 1. If the eccentricity is equal to zero, that means it is a circle. In Kepler's time, the extremes of planetary eccentricity were Venus, 0.007, and Mercury, 0.2. The eccentricity is what makes an ellipse different from a circle.

Figure 5.19 Ellipse



The important components of an ellipse are as follows: semi-major axis a, semi-minor axis b, semi-latus rectum p, the center of the ellipse, and its two foci marked by large dots. For θ = 0°, r = r_{min} and for θ = 180°, r = r_{max}.

$$r = \frac{p}{1 + \Sigma * cos\theta}$$

where: p is the semi-latus rectum, Σ is the eccentricity, r is the distance between the planet and the sun, and θ is the angle between the position of the planet and its most direct route to the Sun. The minimum distance occurs when the angle is 0. The maximum distance occurs when the angle is 180 degrees. These values are important because the equation for eccentricity is:

$$\Sigma = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

The semi-major axis, a, can be found using: $a = \frac{p}{1 - \Sigma^2}$

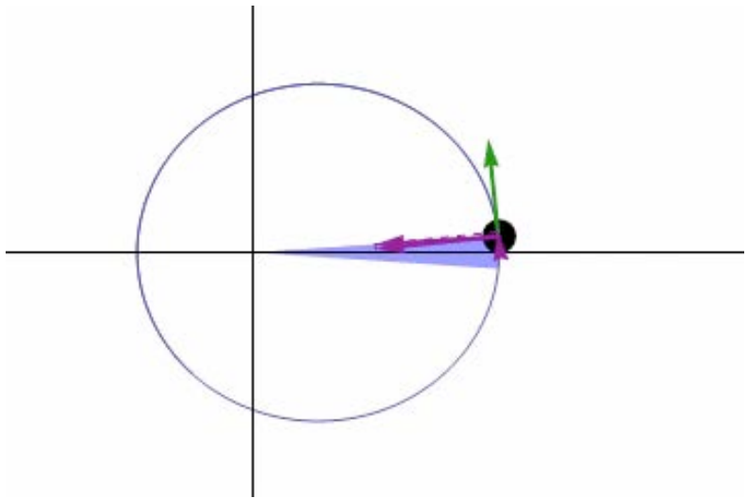
The semi-minor axis, b, can be found using: $b = \frac{p}{\sqrt{1 - \Sigma^2}}$

The area of an ellipse is found with the following equation: $A = \pi * a * b$

Kepler's Second Law

The second law of planetary motion states that in an amount of time, t, a line from the planet to the Sun will sweep out a triangle

Figure 5.20 The Second Law



The planet moves faster near the Sun so that the same area is swept out in a given time as it would be at larger distances, where the planet moves more slowly. The green arrow represents the planet's velocity, and the purple arrows represent the force on the planet.

having a base of r and a height of $r * d\theta$ (*Figure 5.20*). Therefore, the area of this triangle is $dA = \frac{1}{2} * r * rd\theta$ and the ratio of the area of this triangle to the time elapsed is $\frac{dA}{dt} = \frac{1}{2} * r^2 * \frac{d\theta}{dt}$. As the planet moves closer to the Sun, it speeds up. This allows the triangle to have an equal area in an equal amount of time regardless of position of the planet. As we learned in the first section, the area of an ellipse is $\pi * a * b$. Therefore, the period (P) of the ellipse satisfies:

$$\pi * a * b = P * \frac{1}{2} * r^2 * \theta$$

$$r^2\theta = n * a * b$$

where θ is the angular velocity with respect to time, and n is the mean motion of the planet around the Sun.

Kepler's Third Law

Kepler's third law describes the relationship between the distance of the planets from the Sun, and their orbits period. $P^2 \propto a^3$, with a constant of proportionality $\frac{P_{planet}^2}{a_{planet}^3} = \frac{P_{earth}^2}{a_{earth}^3} = 1\frac{yr}{AU}$

where: yr is a year, and an AU is an astronomical unit. In the case of a circular orbit, the proportionality constant is as follows:

$$\frac{4\pi^2}{T^2} = \frac{GM}{R^3}$$

where: T is the period, G is the **gravitational constant**, and R is the distance between the center of mass of the two bodies.

How does Newton Relate to Kepler?

Newton derived his theory of the acceleration of a planet from Kepler's first and second laws. Newton theorized that the direction of a planet is always towards the Sun. In addition, the magnitude of the acceleration is inversely proportional to the square of its distance from the Sun. From this, Newton defined the force acting on a planet as the product of its mass and acceleration. Therefore,

by Newton's law, every planet is attracted to the Sun, and the force acting on a planet is directly proportional to the mass and inversely proportional to the square of its distance from the Sun.

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Tangent Vectors and Normal Vectors

A vector is normal to another vector if the intersection of the two form a 90 degree angle at the tangent point.

KEY POINTS

- In order for one vector to be tangent to another vector, the intersection needs to be exactly 90 degrees. On a curve or an uneven object, each point will have a unique normal vector.
- If you want to check whether two vectors are normal to each other, you can find the dot product of the two and make sure it equals zero.
- If you want to find out exactly what the angle between the two vectors is, you can use the following equation, which also employs the dot product: $a \bullet b = |(a)||(b)|(cos)\theta$.
- In order to find the tangent vector to another vector or object, just take the derivative of the reference vector.

In order for a vector to be normal to an object or vector, it must be **perpendicular** with the directional vector of the **tangent** point. The intersection formed by the two objects must be a right angle.

Normal Vectors

An object is normal to another object if it is perpendicular to the point of reference. That means that the intersection of the two objects forms a right angle. Usually, these vectors are denoted as 'n', as seen in *Figure 5.22*.

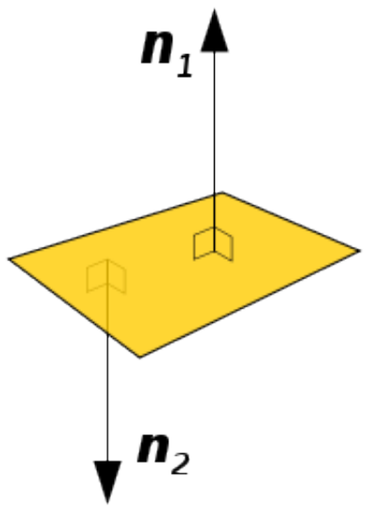

Figure 5.22 Normal Vector
These vectors are normal to the plane because the intersection between them and the plane makes a right angle.

Not only can vectors be 'normal' to objects, but planes can also be normal, as in *Figure 5.21*. This plane is normal to the point on the sphere to which it is tangent. Each point on the sphere will have a unique normal plane, but this will be covered further in another atom. In this atom we are going to focus on vectors.

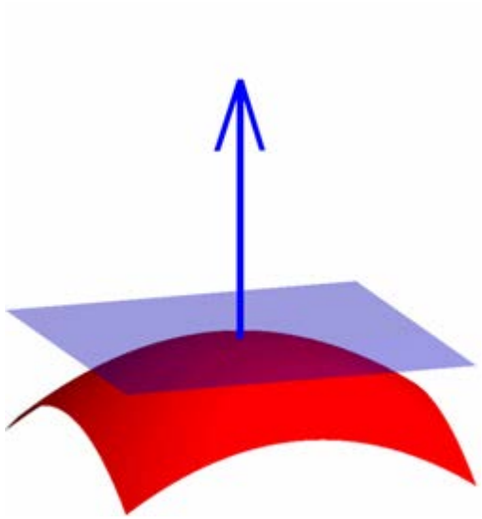

Figure 5.21 Normal Plane
A plane can be determined as normal to the object if the directional vector of the plane makes a right angle with the object at its tangent point.

Dot Product

As we covered in another atom, one of the manipulations of vectors is called the Dot Product. When you take the dot product of two vectors, your answer is in the form of a single value, not a vector. In order for two vectors to be normal to each other, the dot product has to be zero.

$$a \bullet b = 0$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$
$$a \bullet b = |a||b|cos\theta$$

Tangent Vectors

Tangent vectors are almost exactly like normal vectors, except they are tangent instead of normal to the other vector or object. These vectors can be found by obtaining the derivative of the reference vector, r(t):

$$r(t) = f(t)i + g(t)j + h(t)k$$
$$r'(t) = f'(t)i + g'(t)j + h'(t)k$$

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Section 3

Partial Derivatives

Functions of Several Variables

Limits and Continuity

Partial Derivatives

Tangent Planes and Linear Approximations

The Chain Rule

Directional Derivatives and the Gradient Vector

Maximum and Minimum Values

Lagrange Multipliers

Optimization in Several Variables

Applications of Minima and Maxima in Functions of Two Variables

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Functions of Several Variables

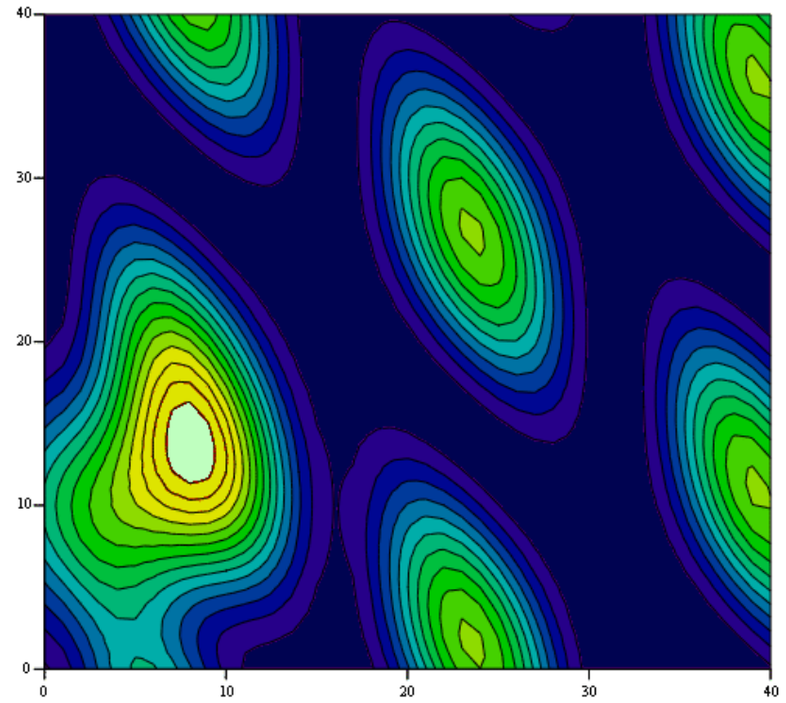
Multivariable calculus is the extension of calculus in one variable to calculus in more than one variable.

KEY POINTS

- Multivariable calculus can be applied to analyze deterministic systems that have multiple degrees of freedom.
- Unlike a single variable function f(x), for which the limits and continuity of the function need to be checked as x varies on a line (x-axis), multivariable functions have infinite number of paths approaching a single point.
- In multivariable calculus, gradient, Stokes', divergence, and Green theorems are specific incarnations of a more general theorem: the generalized Stokes' theorem.

Multivariable calculus (also known as multivariate calculus) is the extension of calculus in one variable to calculus in more than one variable (*Figure 5.23*): the differentiated and integrated functions involve multiple variables, rather than just one. Multivariable calculus can be applied to analyze **deterministic** systems that have multiple degrees of freedom. Functions with independent variables corresponding to each of the degrees of freedom are often used to model these systems, and multivariable calculus provides tools for characterizing the system dynamics.

Figure 5.23 A Scalar Field



A scalar field shown as a function of (x,y). Extensions of concepts used for single variable functions may require caution.

Multivariable calculus is used in many fields of natural and social science and engineering to model and study high-dimensional systems that exhibit deterministic behavior. Non-deterministic, or stochastic, systems can be studied using a different kind of mathematics, such as stochastic calculus. Quantitative analysts in finance also often use multivariate calculus to predict future trends in the stock market.

As we will see, multivariable functions may yield counter-intuitive results when applied to limits and continuity. Unlike a single variable function f(x), for which the limits and continuity of the function needs to be checked as x varies on a single line (x-axis), multivariable functions have infinite number of paths approaching a single point. Likewise, the path taken to evaluate a derivative or integral should always be specified when multivariable functions are involved.

We have also studied theorems linking derivatives and integrals of single variable functions. The theorems we learned are gradient theorem, Stokes' theorem, **divergence** theorem, and Green's theorem. In a more advanced study of multivariable calculus, it is seen that these four theorems are specific incarnations of a more general theorem, the generalized Stokes' theorem, which applies to the integration of differential forms over manifolds.

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Limits and Continuity

A study of limits and continuity in multivariable calculus yields counter-intuitive results not demonstrated by single-variable functions.

KEY POINTS

- The function $f(x,y) = \frac{x^2 y}{x^4 + y^2}$ has different limit values at the origin, depending on the path taken for the evaluation.
- Continuity in each argument does not imply multivariate continuity.
- When taking different paths toward the same point yields different values for the limit, the limit does not exist.

A study of **limits** and **continuity** in multivariable calculus yields many counter-intuitive results not demonstrated by single-variable functions (*Figure 5.24*). For example, there are **scalar functions** of two variables with

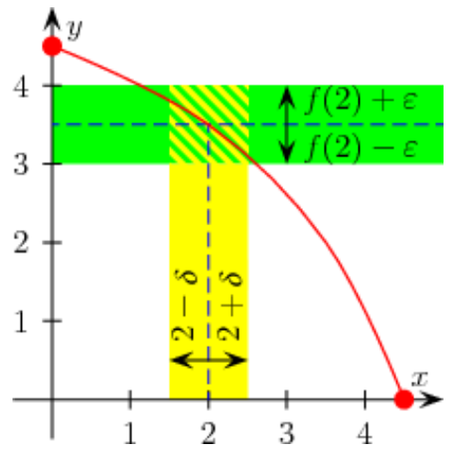

Figure 5.24
Continuity
Continuity in single variable function as shown is rather obvious. However, continuity in multivariable functions yields many counter-intuitive results.

points in their domain which give a particular limit when approached along any arbitrary line, yet give a different limit when approached along a parabola. For example, the function $f(x, y) = \frac{x^2 y}{x^4 + y^2}$ approaches zero along any line through the origin.

However, when the origin is approached along a parabola $y = x^2$, it has a limit of 0.5. Since taking different paths toward the same point yields different values for the limit, the limit does not exist.

Continuity in each argument does not imply multivariate continuity: For instance, in the case of a real-valued function with two real-valued parameters, f(x,y), continuity of f in x for fixed y and continuity of f in y for fixed x does not imply continuity of f. As an example, consider

$$f(x, y) = \begin{cases} \frac{y}{x} - y & \text{if } 1 \geq x > y \geq 0 \\ \frac{x}{y} - x & \text{if } 1 \geq y > x \geq 0 \\ 1 - x & \text{if } x = y > 0 \\ 0 & \text{else}. \end{cases}$$

It is easy to check that all real-valued functions (with one real-valued argument) that are given by $f_y(x) := f(x,y)$ are continuous in x (for any fixed y). Similarly, all f_x are continuous as f is symmetric with regards to x and y. However, f itself is not continuous as can be seen by considering the squence f(1/n,1/n) (for natural n) which should converge to f(0,0)=0 if f was continuous. However, lim f(1/n, 1/n) = 1.

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Partial Derivatives

A partial derivative of a function of several variables is its derivative with respect to a single variable, with the others held constant.

KEY POINTS

- The partial derivative of a function f with respect to the variable x is variously denoted by f'_x, $f_{,x}$, $\partial_x f$, or $\frac{\partial f}{\partial x}$.
- To every point on this surface describing a multi-variable function, there is an infinite number of tangent lines. Partial differentiation is the act of choosing one of these lines and finding its slope.
- As an ordinary derivative, partial derivatives are defined in limit:

$$\frac{\partial}{\partial a_i} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

A partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary). Partial derivatives are used in vector calculus and **differential geometry**. The partial derivative of a function f with respect to the variable x is variously denoted by f'_x, $f_{,x}$, $\partial_x f$, or $\frac{\partial f}{\partial x}$.

Suppose that f is a function of more than one variable. For instance, $z = f(x, y) = x^2 + xy + y^2$. The graph of this function defines a surface

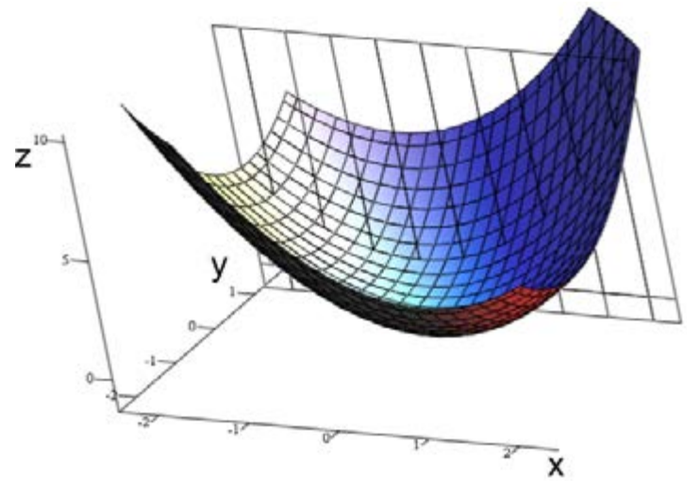

Figure 5.25 A graph of $z = x^2 + xy + y^2$. For the partial derivative at (1, 1, 3) that leaves y constant, the corresponding tangent line is parallel to the xz-plane.

in Euclidean space (*Figure 5.25*). To every point on this surface, there is an infinite number of tangent lines. Partial differentiation is the act of choosing one of these lines and finding its slope. Usually, the lines of most interest are those which are parallel to the xz-plane and those which are parallel to the yz-plane (which result from holding either y or x constant, respectively.)

To find the slope of the line tangent to the function at P(1, 1, 3) that is parallel to the xz-plane, the y variable is treated as constant. The graph and this plane are shown on the right. On the graph in *Figure 5.26*, we show the way the function looks on the plane y = 1. By finding the derivative of the equation while assuming that y is a constant, the slope of f at the point (x, y, z) is found to be:

$\frac{\partial z}{\partial x} = 2x + y$. So at (1, 1, 3), by substitution, the slope is 3. Therefore $\frac{\partial z}{\partial x} = 3$ at the point (1, 1, 3). That is, the partial derivative of z with respect to x at (1, 1, 3) is 3.

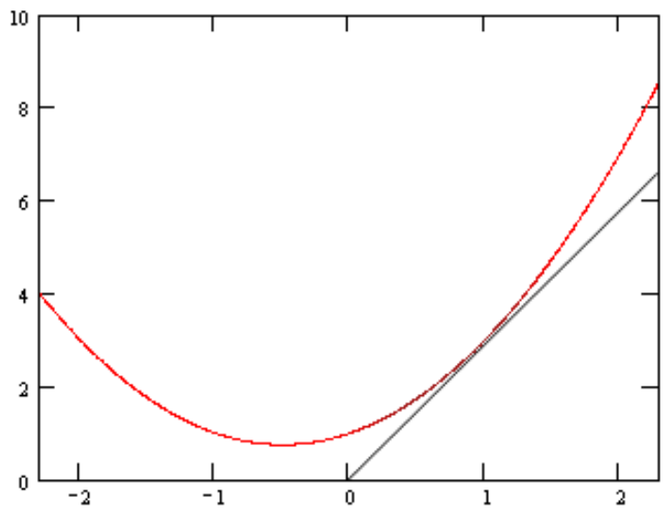

Figure 5.26 y=1

A slice of the graph in Fig 5.17 at y= 1

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Formal Definition

Like ordinary derivatives, the partial derivative is defined as a limit. Let U be an open subset of R^n and $f : U \to R$ a function. The partial derivative of f at the point $a = (a_1,..., a_n) \in U$ with respect to the i-th variable a_i is defined as

$$\frac{\partial}{\partial a_i} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}.$$

Tangent Planes and Linear Approximations

The tangent plane to a surface at a given point is the plane that "just touches" the surface at that point.

KEY POINTS

- For a surface given by a differentiable multivariable function z=f(x,y), the equation of the tangent plane at (xo,yo,zo) is given as $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.
- Since a tangent plane is the best approximation of the surface near the point where the two meet, the tangent plane can be used to approximate the surface near the point.
- The plane describing the linear approximation for a surface described by z=f(x,y) is given as $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

The tangent line (or simply the tangent) to a plane curve at a given point is the straight line that "just touches" the curve at that point. Similarly, the tangent plane to a surface at a given point is the plane that "just touches" the surface at that point (*Figure 5.27*). The concept of a tangent is one of the most fundamental notions in **differential geometry** and has been extensively generalized.

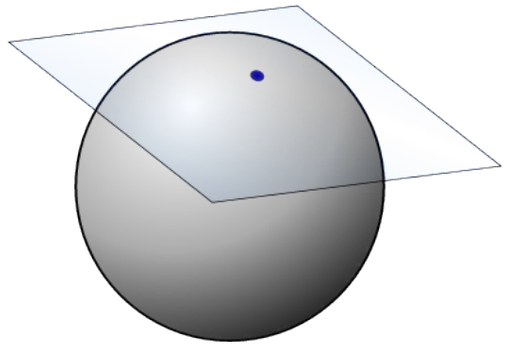

Figure 5.27 Tangent Plane to a Sphere

The tangent plane to a surface at a given point is the plane that "just touches" the surface at that point.

Equations

When the curve is given by y = f(x) the slope of the tangent is $\frac{dy}{dx}$, so by the point–slope formula the equation of the tangent line at (x_0, y_0) is $\frac{dy}{dx}(x_0, y_0) \cdot (x - x_0) - (y - y_0)$, where (x, y) are the coordinates of any point on the tangent line, and where the derivative is evaluated at $x=x_0$.

The tangent plane to a surface at a given point p is defined in an analogous way to the tangent line in the case of curves. It is the best approximation of the surface by a plane at p, and can be obtained as the limiting position of the planes passing through 3 distinct points on the surface close to p as these points converge to p. For a surface given by a differentiable multivariable function z=f(x,y), the equation of the tangent plane at (x_0,y_0,z_0) is given as

$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$, where ($x_o$,$y_o$,$z_o$) is a point on the surface. Note the similarity of the equations for tangent line and tangent plane.

Linear Approximation

Since a tangent plane is the best approximation of the surface near the point where the two meet, tangent plane can be used to approximate the surface near the point. The approximation works well as long as the point (x,y,z) under consideration is close enough to (x_o,y_o,z_o), where the tangent plane touches the surface. The plane describing the linear approximation for a surface described by z=f(x,y) is given as

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

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The Chain Rule

For a function U with two variables x and y, the chain rule is given as $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}$.

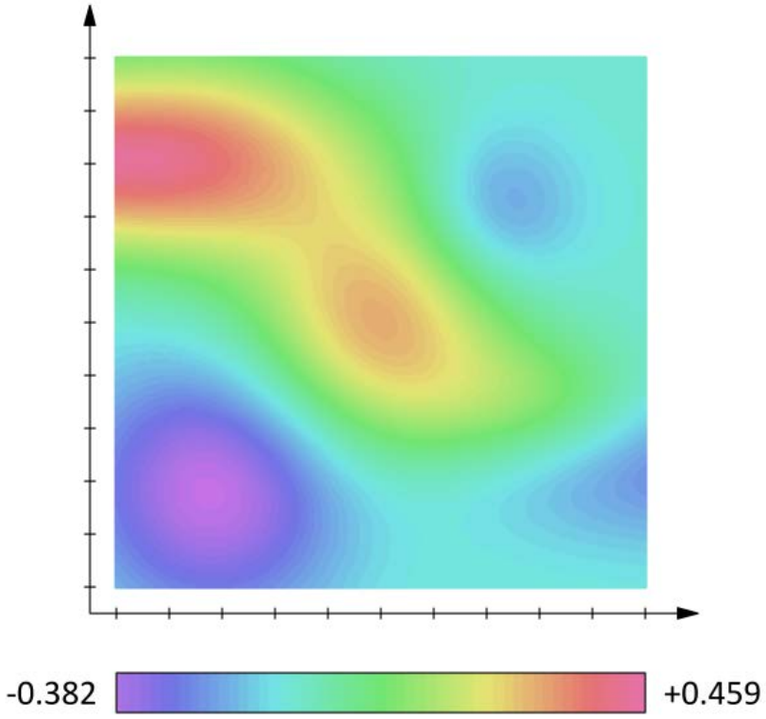
KEY POINTS

- The chain rule can be easily generalized to functions with more than two variables.
- For a single variable functions, the chain rule is a formula for computing the derivative of the composition of two or more functions. For example, the chain rule for f ◦ g (x) ≡ f [g (x)] is $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.
- The chain rule can be used when we want to calculate the rate of change of the function U(x,y) as a function of time t, where x=x(t) and y=y(t).

The chain rule is a formula for computing the derivative of the composition of two or more functions. That is, if f is a function and g is a function, then the chain rule expresses the derivative of the composite function f ◦ g (x) ≡ f [g (x)] in terms of the derivatives of f and g. For example, the chain rule for f ◦ g is $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

The chain rule above is for single variable functions f(x) and g(x). However, the chain rule can be generalized to functions with multiple variables. For example, consider a function U with two variables x and y: U=U(x,y) (*Figure 5.28*). U could be electric potential energy at a location (x,y). The motion of a test charge on the x-y plane can be described by x=x(t), y=y(t), where t is a parameter representing time t. What we want to calculate is the rate of change of the potential energy U as a function of time t. Assuming x=x(t), y=y(t), and U=U(x,y) are all differentiable at t and (x,y), the chain rule is given as $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}$. This relation can be easily generalized for functions with more than two variables.

Figure 5.28 Scalar Field



The chain rule can be used to take derivatives of multivariable functions with respect to a parameter.

Example: For $z = (x^2 + xy + y^2)^{1/2}$ where x=x(t) and y=y(t), express $\frac{dz}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dz}{dt} = \frac{d}{dt}(x^2 + xy + y^2)^{1/2}$$

$$= \frac{1}{2}(x^2 + xy + y^2)^{-1/2}\frac{d}{dt}(x^2 + xy + y^2)$$

$$= \frac{1}{2}(x^2 + xy + y^2)^{-1/2}\left[\frac{d}{dt}(x^2) + \frac{d}{dt}(xy) + \frac{d}{dt}(y^2)\right]$$

$$= \frac{1}{2}(x^2 + xy + y^2)^{-1/2}\left[2x\frac{dx}{dt} + y\frac{dx}{dt} + x\frac{dy}{dt} + 2y\frac{dy}{dt}\right]$$

$$= \frac{(x + \frac{1}{2}y)\frac{dx}{dt} + (y + \frac{1}{2}x)\frac{dy}{dt}}{\sqrt{x^2 + xy + y^2}}.$$

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Directional Derivatives and the Gradient Vector

The directional derivative represents the instantaneous rate of change of the function, moving through **x** with a velocity specified by **v**.

KEY POINTS

- The directional derivative is defined by the limit $\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$.
- If the function f is differentiable at x, then the directional derivative exists along any vector v, and one gets $\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$.
- Many of the familiar properties of the ordinary derivative hold for the directional derivative.

The directional derivative of a multivariate differentiable function along a given vector v at a given point x intuitively represents the instantaneous rate of change of the function, moving through x with a velocity specified by v. It therefore generalizes the notion of a partial derivative, in which the rate of change is taken along one of the coordinate curves, all other coordinates being constant.

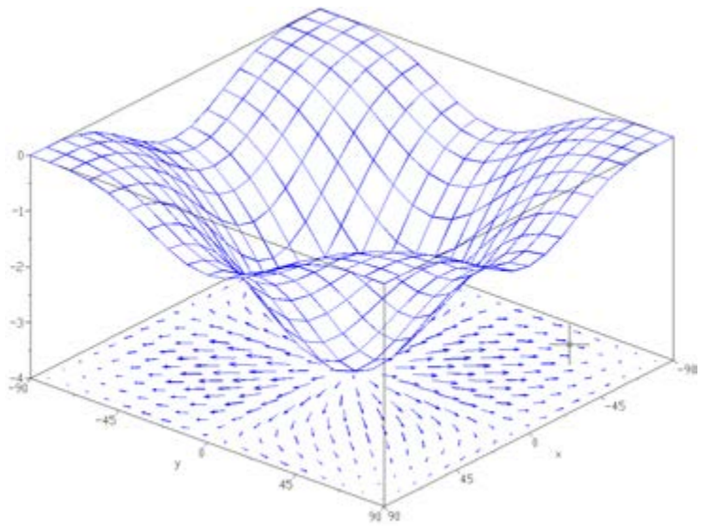
Definition

The directional derivative of a scalar function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ along a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is the function defined by the limit

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}.$$

If the function f is differentiable at x, then the directional derivative exists along any vector v, and one has $\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$, where the $\nabla f(\mathbf{x})$ is the gradient vector and $\cdot$ is the dot product. At any point x, the directional derivative of f intuitively represents the rate of change of f with respect to time when it is moving at a speed and direction given by v at the point x (*Figure 5.29*).

Figure 5.29 Gradient of a Function



The gradient of the function $f(x, y) = -(cosx)^2 + (cosy)^2$ depicted as a projected vector field on the bottom plane. Directional derivative represents the rate of change of the function along any direction specified by v.

Properties

Many of the familiar properties of the ordinary derivative hold for the directional derivative.

1. The sum rule: $\nabla_v(f + g) = \nabla_v f + \nabla_v g$.

2. The constant factor rule: For any constant c, $\nabla_v(cf) = c\nabla_v f$.

3. The product rule (or Leibniz rule): $\nabla_v(fg) = g\nabla_v f + f\nabla_v g$.

4. The chain rule: If g is differentiable at p and h is differentiable at g(p), then $\nabla_v h \circ g(p) = h'(g(p))\nabla_v g(p)$.

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Maximum and Minimum Values

The second partial derivative test is a method used to determine whether a critical point is a local minimum, maximum, or saddle point.

KEY POINTS

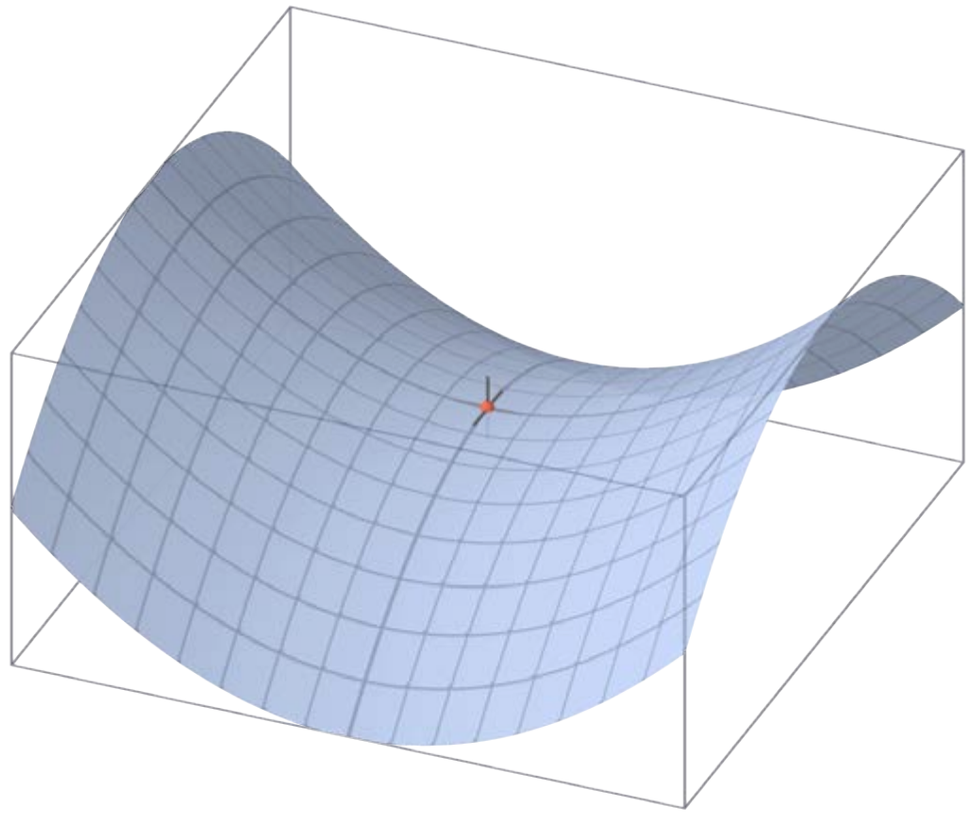
- For a function of two variables, the second partial derivative test is based on the sign of $M(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$ and $f_{xx}(a,b)$, where (a,b) is a critical point.
- There are substantial differences between the functions of one variable and the functions of more than one variable in the identification of global extrema.
- The maximum and minimum of a function, known collectively as extrema, are the largest and smallest values that the function takes at a point either within a given neighborhood (local or relative extremum) or on the function domain in its entirety (global or absolute extremum).

The maximum and minimum of a function, known collectively as extrema, are the largest and smallest values that the function takes at a point either within a given neighborhood (local or relative extremum) or on the function domain in its entirety (global or absolute extremum).

Finding Maxima and Minima of Multivariable Functions

The second partial derivative test is a method in multivariable calculus used to determine whether a **critical point** (a,b,...) of a function f(x,y,...) is a local minimum, maximum, or saddle point (*Figure 5.30*).

Figure 5.30 Saddle Point



A saddle point on the graph of $z = x^2 - y^2$ (in red).

For a function of two variables, suppose that $M(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$.

1. If $f_{xx}(a,b) > 0$, then (a,b) is a local minimum of f.
2. If $f_{xx}(a,b) < 0$, then (a,b) is a local maximum of f.
3. If $M(a,b) < 0$, then (a,b) is a saddle point of f.
4. If $M(a,b) = 0$, then the second derivative test is inconclusive.

There are substantial differences between functions of one variable and functions of more than one variable in the identification of global extrema. For example, if a bounded differentiable function f defined on a closed interval in the real line has a single critical point, which is a local minimum, then it is also a global minimum (use the intermediate value theorem and Rolle's theorem). In two and more dimensions, this argument fails, as the function $f(x,y) = x^2 + y^2(1-x)^3,\ x,y \in \mathrm{R}$ shows. Its only critical point is at (0,0), which is a local minimum with $f(0,0) = 0$. However, it cannot be a global one, because $f(4,1) = -11$.

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Lagrange Multipliers

The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints.

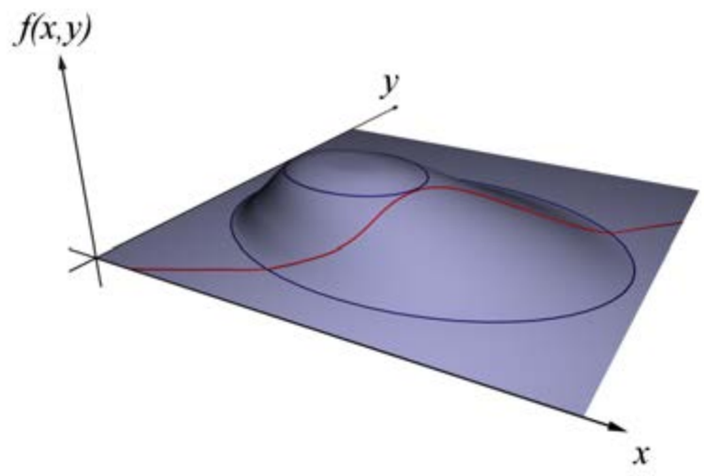
KEY POINTS

- To maximize f(x,y) subject to g(x,y)=c, we introduce a new variable λ, called a Lagrange multiplier, and study the Lagrange function (or Lagrangian) defined by $\Lambda(x, y, \lambda) = f(x, y) + \lambda \cdot \Big(g(x, y) - c\Big)$.
- Only when the contour line for g = c meets the contour lines of f tangentially do we not increase or decrease the value of f — that is, when the contour lines touch but do not cross.
- Solve $\nabla_{x,y,\lambda} \Lambda(x, y, \lambda) = 0$, and we find a necessary condition for extrema under the given constraint.

In mathematical optimization, the method of Lagrange multipliers (named after Joseph Louis Lagrange) is a strategy for finding the local maxima and minima of a function subject to equality constraints.

For instance (*Figure 5.31*), consider the following optimization problem: Maximize f(x,y) subject to g(x,y)=c. We need both f and g to have continuous first partial derivatives. We introduce a new variable (λ) called a Lagrange multiplier, and study the Lagrange function (or Lagrangian) defined by $\Lambda(x, y, \lambda) = f(x, y) + \lambda \cdot \Big(g(x, y) - c\Big)$, where the λ term may be either added or subtracted. If $f(x_o, y_o)$ is a maximum of f(x,y) for the original constrained problem, then there exists λ_0 such that (x_0, y_0, λ_0) is a stationary point for the Lagrange function (stationary points are those points where the partial derivatives of Λ are zero, i.e. $\nabla\Lambda = 0$. However, not all stationary points yield a solution of the original problem. Thus, the method of Lagrange multipliers yields a necessary condition for optimality in constrained problems. Sufficient conditions for a minimum or maximum also exist.

Figure 5.31 Maximizing f(x,y)



Find x and y to maximize f(x,y) subject to a constraint (shown in red) g(x,y)=c.

Introduction

One of the most common problems in calculus is that of finding maxima or minima (in general, "extrema") of a function, but it is often difficult to find a closed form for the function being extremized. Such difficulties often arise when one wishes to maximize or minimize a function subject to fixed outside conditions or constraints. The method of Lagrange multipliers is a powerful tool for solving this class of problems without the need to explicitly solve the conditions and use them to eliminate extra variables.

Consider the two-dimensional problem introduced above: Maximize f(x,y) subject to g(x,y)=c. We can visualize contours of f given by $f(x,y) = d$ for various values of d, and the contour of g given by g (x, y) = c. Suppose we walk along the contour line with g = c. In general, the contour lines of f and g may be distinct, so following the contour line for g = c, one could intersect with or cross the contour lines of f. This is equivalent to saying that while moving along the contour line for g = c, the value of f can vary. Only when the contour line for g = c meets contour lines of f tangentially do we not increase or decrease the value of f--that is, when the contour lines touch but do not cross.

The contour lines of f and g touch when the tangent vectors of the contour lines are parallel. Since the gradient of a function is perpendicular to the contour lines, this is the same as saying that the gradients of f and g are parallel. Thus, we want points (x,y) where g(x,y) = c and $\nabla_{x,y} f = -\lambda \nabla_{x,y} g$, where $\nabla_{x,y} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $\nabla_{x,y} g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ are the respective gradients. The constant is required because, although the two gradient vectors are parallel, the magnitudes of the gradient vectors are generally not equal.

To incorporate these conditions into one equation, we introduce an auxiliary function, $\Lambda(x, y, \lambda) = f(x, y) + \lambda \cdot \Big(g(x, y) - c\Big)$, and solve $\nabla_{x,y,\lambda} \Lambda(x, y, \lambda) = 0$. This is the method of Lagrange multipliers. Note that $\nabla_{\lambda} \Lambda(x, y, \lambda) = 0$ implies $g(x, y) = c$.

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Optimization in Several Variables

To solve an optimization problem, formulate the function f(x,y,...) to be optimized and find all critical points first.

KEY POINTS

- Mathematical optimization is the selection of a best element (with regard to some criteria) from some set of available alternatives.
- An optimization process that involves only a single variable is rather straightforward. After finding out the function f(x) to be optimized, local maxima or minima at critical points can easily be found. End points may have maximum/minimum values as well.
- For a rectangular cuboid shape, given the fixed volume, a cube is the geometric shape that minimizes the surface area.

Mathematical **optimization** is the selection of a best element (with regard to some criteria) from some set of available alternatives. An optimization process that involves only a single variable is rather straightforward. After finding out the function f(x) to be optimized, local maxima or minima at critical points can be easily found. (Of course, end points may have maximum/minimum values as well.)

The same strategy applies for optimization with several variables. In this atom, we will solve a simple example to see how optimization involving several variables can be achieved.

Cardboard Box with a Fixed Volume

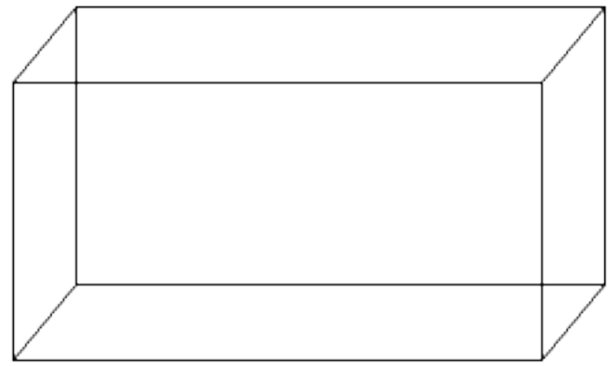

Figure 5.32
Rectangular Cuboid
Mathematical Optimization can be used to solve problems that involve finding the right size of a volume such as a cuboid.

A packaging company needs cardboard boxes in rectangular **cuboid** shape with a given volume (=1000 cm^3) and would like to minimize the material cost for the boxes. What should be the dimensions x, y, z of a box?

First of all, the material cost would be proportional to the surface area S of the cuboid (*Figure 5.32*). Therefore, the goal of the optimization is to minimize a function: $S(x, y, z) = 2(xy + yz + zx)$. The constraint in the case is that the volume is fixed: $V = xyz = 1000$.

We will first remove z from S(x,y,z). We can do this by using the constraint $z = \frac{1000}{xy}$. Inserting the expression for z in S(x,y,z), we get $S(x, y, z) = 2\left(xy + \frac{1000}{x} + \frac{1000}{y}\right)$. To find the critical points,

$$\frac{\partial S}{\partial x} = 2(y - \frac{1000}{x^2}) = 0 \Rightarrow y = \frac{1000}{x^2} \quad (1)$$

$$\frac{\partial S}{\partial y} = 2(x - \frac{1000}{y^2}) = 0 \Rightarrow x = \frac{1000}{y^2} \quad (2)$$

Using the expression for y in Eq (1), Eq (2) becomes $x^3 = 1000$. Therefore, we find that $x = y = z = 10$. The box that minimizes the material cost (while maintaining the volume) should be a cube.

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Applications of Minima and Maxima in Functions of Two Variables

Finding extrema can be a challenge with regard to multivariable functions, requiring careful calculation.

KEY POINTS

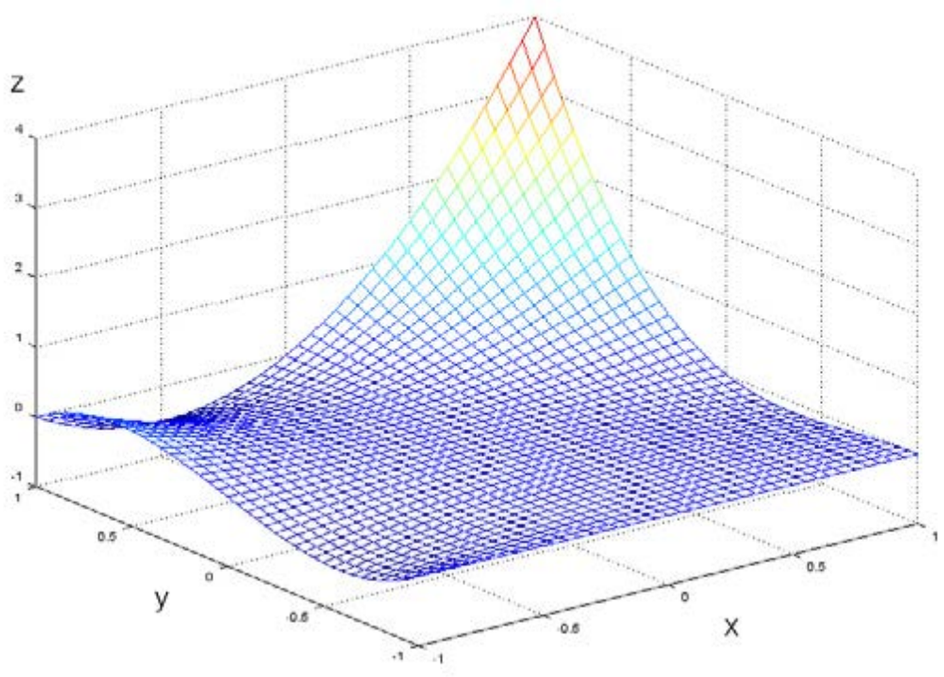
- The second derivative test is a criterion for determining whether a given critical point of a real function of one variable is a local maximum or a local minimum using the value of the second derivative at the point.
- To find minima/maxima for functions with two variables, we must first find the first partial derivatives with respect to x and y of the function.
- The function $z = f(x, y) = (x + y)(xy + xy^2)$ has a saddle point at (0,-1) and (1,-1), and a local maximum at (3/8, -3.4).

We have learned how to find the minimum and maximum in **multivariable** functions. As previously mentioned, finding extrema can be a challenge with regard to multivariable functions. In particular, we learned about the second derivative test, which is a criterion for determining whether a given **critical point** of a real function of one variable is a local maximum or a local minimum,

using the value of the second derivative at the point. In this atom, we will find extrema for a function with two variables.

Example: Find and label the critical points of the following function: $z = f(x, y) = (x + y)(xy + xy^2)$ (*Figure 5.33*)

Figure 5.33 Plot of $z = (x + y)(xy + xy^2)$



Plot of $z = (x + y)(xy + xy^2)$. The maxima and minima of this plot cannot be found without extensive calculation.

To solve this problem we must first find the first partial derivatives with respect to x and y of the function.

$$\frac{\partial z}{\partial x} = y(2x + y)(y + 1)$$

$$\frac{\partial z}{\partial y} = x\left(3y^2 + 2y(x + 1) + x\right)$$

Looking at $\frac{\partial z}{\partial x} = 0$ we see that y must equal 0, −1 or -2x.

We plug the first solution y = 0 into the next equation, and get $\frac{\partial z}{\partial y} = x\left(3y^2 + 2y(x + 1) + x\right) = x^2$. There were other possibilities for y, so for y = -1 we have $\frac{\partial z}{\partial y} = x\left(3 - 2(x + 1) + x\right) = x(1 - x) = 0$. So x must be equal to 1 or 0. Finally for y = -2x: $\frac{\partial z}{\partial y} = x\left(3(-2x)^2 + 2(-2x)(x + 1) + x\right) = x^2(8x - 3) = 0$. So x must equal 0 or $\frac{3}{8}$ for y = 0 and $-\frac{3}{4}$, respectively.

Let's list all the critical values now: $(x, y) \in (0,0), (0, -1), (1, -1), \left(\frac{3}{8}, -\frac{3}{4}\right)$. Now we have to label the critical values using the second derivative test:

$$D = f_{xx}(a, b)f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2 = 2y(y + 1)(2(3y + x + 1))x - (3y^2 + y(4x + 2) + 2x)^2$$

Plugging in all the different critical values we found to label them, we have

$$D(0,0) = 0; \quad D(0,-1) = -1; \quad D(1,-1) = -1; \quad D\left(\frac{3}{8}, -\frac{3}{4}\right) = 0.210938$$

We can now label some of the points. At (0, −1) and (1, −1), f(x, y) has a saddle point; at $\left(\frac{3}{8}, -\frac{3}{4}\right)$ it has a local maximum, since $f_{xx} = -\frac{3}{8} < 0$. At the remaining point we need higher order tests to find out what exactly the function is doing.

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Section 4

Multiple Integrals

Double Integrals Over Rectangles

Iterated Integrals

Double Integrals Over General Regions

Double Integrals in Polar Coordinates

Triple Integrals in Cylindrical Coordinates

Triple Integrals in Spherical Coordinates

Triple Integrals

Change of Variables

Applications of Multiple Integrals

Center of Mass and Inertia

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Double Integrals Over Rectangles

For a rectangular region S defined by x in [a,b] and y in [c,d], the double integral of a function f(x,y) in this region is given as

$$\iint_S f(x,y)dxdy = \int_a^b (\int_c^d f(x,y)dy)dx = \int_c^d (\int_a^b f(x,y)dx)dy.$$

KEY POINTS

- The multiple integral is a type of definite integral extended to functions of more than one real variable--for example, f(x, y) or f(x, y, z). Integrals of a function of two variables over a region in R^2 are called double integrals.
- The double integral of a positive function of two variables represents the volume of the region between the surface defined by the function (on the three dimensional Cartesian plane where z = f(x, y)) and the plane which contains its domain.
- If there are more variables than 3, a multiple integral will yield hypervolumes of multi-dimensional functions.

The multiple integral is a type of definite integral extended to functions of more than one real variable--for example, f(x, y) or f(x, y, z). Integrals of a function of two variables over a region in R^2 are called double integrals. Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function (on the three dimensional Cartesian plane where z = f(x, y)) and the plane which contains its domain. (*Figure 5.34*) The same volume can be obtained via the triple integral—the integral of a function in three variables—of the constant function f(x, y, z) = 1 over the above-mentioned region between the surface and the plane. If there are more variables, a multiple integral will yield **hypervolumes** of multi-dimensional functions.

Figure 5.34 Volume to be Integrated

Double integral as volume under a surface z = x^2 – y^2. The rectangular region at the bottom of the body is the domain of integration, while the surface is the graph of the two-variable function to be integrated.

Double Integrals Over Rectangles

Double integrals over rectangular regions are straightforward to compute in many cases. For a rectangular region S defined by x in [a,b] and y in [c,d], the double integral of a function f(x,y) in this region is given as:

$$\iint_S f(x,y)dxdy = \int_a^b (\int_c^d f(x,y)dy)dx = \int_c^d (\int_a^b f(x,y)dx)dy.$$

Here, we exchanged the order of the integration, assuming that f(x,y) satisfies the conditions to apply **Fubini's theorem**.

Example

Let us assume that we wish to integrate a multivariable function f over a region A.

$A = \{(x,y) \in \mathbf{R}^2 \ : \ 11 \leq x \leq 14 \ ; \ 7 \leq y \leq 10\}, f(x,y) = x^2 + 4y.$

Formulating the double integral $\int_7^{10} \int_{11}^{14} (x^2 + 4y) \ dx \, dy$, we first evaluate the inner integral with respect to x:

$$\int_{11}^{14} (x^2 + 4y) \ dx$$

$$= \left(\frac{1}{3}x^3 + 4yx\right)\Big|_{x=11}^{x=14}$$

$$= \frac{1}{3}(14)^3 + 4y(14) - \frac{1}{3}(11)^3 - 4y(11)$$

$$= 471 + 12y$$

We then integrate the result with respect to y.

$$\int_7^{10} (471 + 12y) \ dy$$

$$= (471y + 6y^2)\Big|_{y=7}^{y=10}$$

$$= 471(10) + 6(10)^2 - 471(7) - 6(7)^2$$

$$= 1719$$

We could have computed the double integral starting from the integration over y. Confirm yourself that the result is the same.

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Iterated Integrals

An iterated integral is the result of applying integrals to a function of more than one variable.

KEY POINTS

- The function f(x,y), if y is considered a given parameter, can be integrated with respect to x as follows: $\int f(x,y)dx$.
- The result is a function of y and therefore its integral can be considered again. If this is done, the result is the iterated integral $\int \left(\int f(x,y)\,dx \right) dy$.
- It is key to note that this is different, in principle, to the multiple integral $\iint f(x,y)\,dx\,dy$.

An iterated integral is the result of applying integrals to a function of more than one variable (for example f(x,y) or f(x,y,z)) in such a way that each of the integrals considers some of the variables as given constants. For example, in the function f(x,y), if y is considered a given parameter, it can be integrated with respect to x, $\int f(x,y)dx$. The result is a function of y and therefore its integral can be considered again. If this is done, the result is the iterated integral $\int \left(\int f(x,y)\,dx \right) dy$.

It is key to note that this is different, in principle, from the multiple integral $\iint f(x,y)\,dx\,dy$ (*Figure 5.35*). A theorem called Fubini's theorem, however, states that they may be equal under very mild conditions. The alternative notation for iterated integrals $\int dy \int f(x,y)\,dx$ is also used. Iterated integrals are computed following the operational order indicated by the parentheses (in the notation that uses them), starting from the innermost integral and working out.

Figure 5.35 Use of an iterated integral
An iterated integral can be used to find the volume of the object in the figure.

Example

For the iterated integral $\int \left(\int (x+y)\,dx \right) dy$, the integral $\int (x+y)\,dx = \frac{x^2}{2} + yx$ is computed first. The result is then used to compute the integral with respect to y.

$$\int (\frac{x^2}{2} + yx)\, dy = \frac{yx^2}{2} + \frac{xy^2}{2}$$

It should be noted, however, that this example omits the constants of integration. After the first integration with respect to x, we would rigorously need to introduce a "constant" function of y. That is, If we were to differentiate this function with respect to x, any terms containing only y would vanish, leaving the original integral. Similarly for the second integral, we would introduce a "constant" function of x, because we have integrated with respect to y. In this way, indefinite integration does not make much sense for functions of several variables. While the antiderivatives of single variable functions differ at most by a constant, the antiderivatives of multivariable functions differ by unknown single-variable terms, which could have a drastic effect on the behavior of the function.

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Double Integrals Over General Regions

Double integrals can be evaluated over the integral domain of any general shape.

KEY POINTS

- If the domain D is normal with respect to the x-axis, and f : D → R is a continuous function, then α(x) and β(x) (defined on the interval [a, b]) are the two functions that determine D.
$$\iint_D f(x,y)\; dx\, dy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy.$$
- Applying this general method, the projection of D onto either the x-axis or the y-axis should be bounded by the two values, a and b.
- For a domain $D = \{(x,y) \in \mathbf{R}^2 \;:\; x \geq 0, y \leq 1, y \geq x^2\}$, we can write the integral over D as
$$\iint_D (x+y)\, dx\, dy = \int_0^1 dx \int_{x^2}^1 (x+y)\, dy.$$

We studied how double integrals can be evaluated over a rectangular region. But there is no reason to limit the domain to a rectangular area. The integral domain can be of any general shape. In this atom, we will study how to formulate such an integral.

This method is applicable to any domain D for which:

- the projection of D onto either the x-axis or the y-axis is bounded by the two values, a and b.
- any line perpendicular to this axis that passes between these two values intersects the domain in an interval whose endpoints are given by the graphs of two functions, α and β.

X-axis

If the **domain** D is normal with respect to the x-axis, and f : D → R is a continuous function, then α(x) and β(x) (defined on the interval [a, b]) are the two functions that determine D. It follows, then, that

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy.$$

y-axis: If D is normal with respect to the y-axis and f : D → R is a continuous function, then α(y) and β(y) (defined on the interval [a, b]) are the two functions that determine D. It follows, then, that

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dy \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx.$$

Example

Consider the following region (please see the graphic in the example): $D = \{(x,y) \in \mathbf{R}^2 \ : \ x \geq 0, y \leq 1, y \geq x^2\}$ (*Figure 5.36*). Calculate $\iint_D (x+y)\,dx\,dy$. This domain is normal with respect to both the x- and y-axes. To apply the formulae, you must first find the functions that determine D and the intervals over which these are defined. In this case the two functions are $\alpha(x) = x^2$ and $\beta(x) = 1$, while the interval is given by the intersections of the functions with x = 0, so the interval is [a, b] = [0, 1] (normality has been chosen with respect to the x-axis for a better visual understanding).

It is now possible to apply the formula

$$\iint_D (x+y)\,dx\,dy = \int_0^1 dx \int_{x^2}^1 (x+y)\,dy = \int_0^1 dx \left[xy + \frac{y^2}{2} \right]_{x^2}^1.$$ (At first the second integral is calculated considering x as a constant). The remaining operations consist of applying the basic techniques of integration: $\int_0^1 \left[xy + \frac{y^2}{2} \right]_{x^2}^1 dx = \int_0^1 \left(x + \frac{1}{2} - x^3 - \frac{x^4}{2} \right) dx = \frac{13}{20}.$

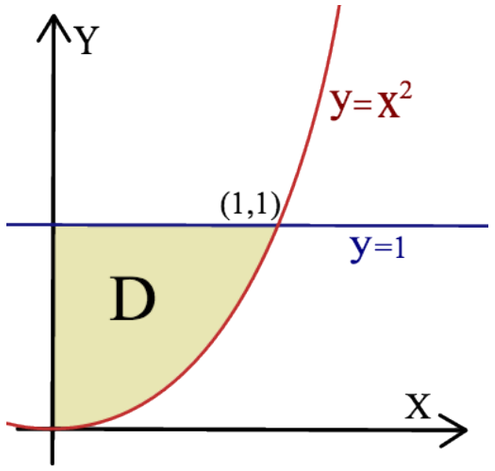

Figure 5.36 Double Integral

Double integral over the normal region D shown in the example.

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Double Integrals in Polar Coordinates

When domain has a cylindrical symmetry and the function has several specific characteristics, apply the transformation to polar coordinates.

KEY POINTS

- The fundamental relation to make the transformation is the following: $f(x, y) \to f(\rho \cos \phi, \rho \sin \phi)$.
- To switch the integral from Cartesian to polar coordinates, the dx dy differentials in this transformation become ρ dρ dφ.
- Once the function is transformed and the domain evaluated, it is possible to define the formula for the change of variables in polar coordinates:
$$\iint_D f(x, y)\, dx\, dy = \iint_T f(\rho \cos \phi, \rho \sin \phi) \rho\, d\rho\, d\phi.$$

In R^2, if the domain has a cylindrical symmetry and the function has several particular characteristics, you can apply the transformation to polar coordinates, which means that the generic points P(x, y) in **Cartesian** coordinates switch to their respective points in polar coordinates (*Figure 5.37*). This allows one to change the shape of the domain and simplify the operations.

Change of variable

The polar coordinates r and φ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$x = r\cos\phi$

$y = r\sin\phi\,.$

The Cartesian coordinates x and y can be converted to polar coordinates r and φ with r ≥ 0 and φ in the interval (−π, π] by

$r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}(\frac{y}{x}).$

The fundamental relation to make the transformation is as follows:
$f(x, y) \to f(\rho\cos\phi, \rho\sin\phi)$.

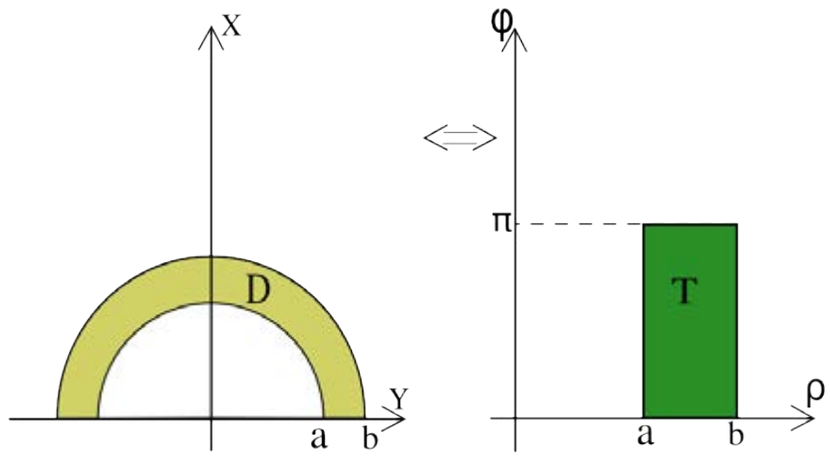
Example 1

Given the function $f(x, y) = x + y$ and applying the transformation, one obtains $f(\rho, \phi) = \rho\cos\phi + \rho\sin\phi = \rho(\cos\phi + \sin\phi)$.

Example 2

Given the function $f(x, y) = x^2 + y^2$, one can obtain $f(\rho, \phi) = \rho^2(\cos^2\phi + \sin^2\phi) = \rho^2$ using the Pythagorean trigonometric identity, which is very useful to simplify this

Figure 5.37 Transformation to Polar Coordinates



This figure illustrates graphically a transformation from cartesian to polar coordinates

operation. Particularly in this case, you can see that the representation of the function f became simpler in polar coordinates. This is the case because the function has a cylindrical symmetry. In general, the best practice is to use the coordinates that match the built-in symmetry of the function.

Integrals in Polar Coordinates

The Jacobian determinant of that transformation is the following:
$\frac{\partial(x, y)}{\partial(\rho, \phi)} = \begin{vmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{vmatrix} = \rho$, which has been obtained by

inserting the partial derivatives of x = ρ cos(φ), y = ρ sin(φ) in the first column with respect to ρ and in the second column with respect to φ, so the (dx dy) differentials in this transformation

becomes (ρ dρ dφ). Once the function is transformed and the domain evaluated, it is possible to define the formula for the change of variables in polar coordinates:

$$\iint_D f(x,y)\,dx\,dy = \iint_T f(\rho\cos\phi, \rho\sin\phi)\rho\,d\rho\,d\phi.$$

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Example

Integrate the function f(x,y) = x over the domain $D = \{x^2 + y^2 \leq 9,\ x^2 + y^2 \geq 4,\ y \geq 0\}$. From $f(x,y) = x \longrightarrow f(\rho,\phi) = \rho\cos\phi$,

$$\iint_D x\,dx\,dy = \iint_T \rho\cos\phi\rho\,d\rho\,d\phi$$

$$= \int_0^{\pi}\int_2^3 \rho^2\cos\phi\,d\rho\,d\phi$$

$$= \int_0^{\pi}\cos\phi\ d\phi\left[\frac{\rho^3}{3}\right]_2^3$$

$$= \left[\sin\phi\right]_0^{\pi}\left(9 - \frac{8}{3}\right) = 0.$$

Triple Integrals in Cylindrical Coordinates

When the function to be integrated has a cylindrical symmetry, it is sensible to integrate using cylindrical coordinates.

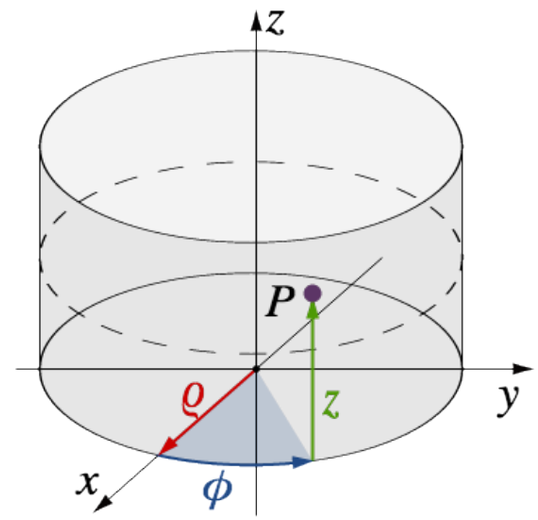
KEY POINTS

- Switching from Cartesian to cylindrical coordinates, the transformation of the function is made by the following relation $f(x, y, z) \to f(\rho \cos \phi, \rho \sin \phi, z)$.
- In switching to cylindrical coordinates, the (dx dy dz) differentials in the integral becomes (ρ dρ dφ dz).
- Therefore, an integral evaluated in Cartesian coordinates can be switched to an integral in cylindrical coordinates as $\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho \cos \phi, \rho \sin \phi, z)\rho\, d\rho\, d\phi\, dz$.

When the function to be integrated has a cylindrical symmetry, it is sensible to change the variables into cylindrical coordinates and then perform integration.

In R^3 the integration on domains with a circular base can be made by the passage in **cylindrical coordinates** (*Figure 5.38*); the transformation of the function is made by the following relation: $f(x, y, z) \to f(\rho \cos \phi, \rho \sin \phi, z)$. The domain transformation can be graphically attained, because only the shape of the base varies, while the height follows the shape of the starting region. Also in switching to cylindrical coordinates, the (dx dy dz) differentials in the integral becomes (ρ dρ dφ dz). (See our atom on "Double integral in polar coordinates.")

Figure 5.38 Cylindrical Coordinates



Cylindrical coordinates are often used for integrations on domains with a circular base.

Example 1

The region is $D = \{x^2 + y^2 \leq 9,\ x^2 + y^2 \geq 4,\ 0 \leq z \leq 5\}$; if the transformation is applied, this region is obtained: $T = \{2 \leq \rho \leq 3,\ 0 \leq \phi \leq 2\pi,\ 0 \leq z \leq 5\}$. Because the z component is unvaried during the transformation, the dx dy dz differentials vary as in the passage in polar coordinates: therefore, they become ρ dρ dφ dz. Finally, it is possible to apply the final formula to cylindrical coordinates:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho \cos \phi, \rho \sin \phi, z)\rho\, d\rho\, d\phi\, dz.$$

This method is convenient in case of cylindrical or conical domains or in regions where it is easy to individuate the z interval and even transform the circular base and the function.

Example 2

The function $f(x, y, z) = x^2 + y^2 + z$ is and as integration domain this cylinder: $D = \{x^2 + y^2 \leq 9,\ -5 \leq z \leq 5\}$. The transformation of D in cylindrical coordinates is the following:
$T = \{0 \leq \rho \leq 3,\ 0 \leq \phi \leq 2\pi,\ -5 \leq z \leq 5\}$, while the function becomes $f(\rho \cos \phi, \rho \sin \phi, z) = \rho^2 + z$. Therefore, the integral becomes

$$\iiint_D (x^2 + y^2 + z)\, dx\, dy\, dz = \iiint_T (\rho^2 + z)\rho\, d\rho\, d\phi\, dz$$

$$= \int_{-5}^{5} dz \int_0^{2\pi} d\phi \int_0^3 (\rho^3 + \rho z)\, d\rho$$

$$= 2\pi \int_{-5}^{5} \left[\frac{\rho^4}{4} + \frac{\rho^2 z}{2} \right]_0^3 dz$$

$$= 2\pi \int_{-5}^{5} \left(\frac{81}{4} + \frac{9}{2} z \right) dz$$

$$= 405\pi.$$

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Triple Integrals in Spherical Coordinates

When the function to be integrated has a spherical symmetry, change the variables into spherical coordinates and then perform integration.

KEY POINTS

- Switching from Cartesian to spherical coodinates, the function is transformed by this relation: $f(x, y, z) \longrightarrow f(\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$.
- For the transformation, the (dx dy dz) differentials in the integral are transformed to (ρ2 sinφ dρ dθ dφ).
- Therefore, an integral evaluated in Cartesian coordinates can be switched to an integral in spherical coordinates as $\iiint_D f(x, y, z)\, dx\, dy\, dz =$ $\iiint_T f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi]$

When the function to be integrated has a spherical symmetry, it is sensible to change the variables into **spherical coordinates** (*Figure 5.39*) and then perform integration.

In R^3 some domains have a spherical symmetry, so it's possible to specify the coordinates of every point of the integration region by two angles and one distance. It's possible to use therefore the passage in spherical coordinates; the function is transformed by this relation: $f(x, y, z) \longrightarrow f(\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$. Points on the z axis do not have a precise characterization in spherical coordinates, so θ can vary between 0 to 2π. The better integration domain for this passage is obviously the sphere.

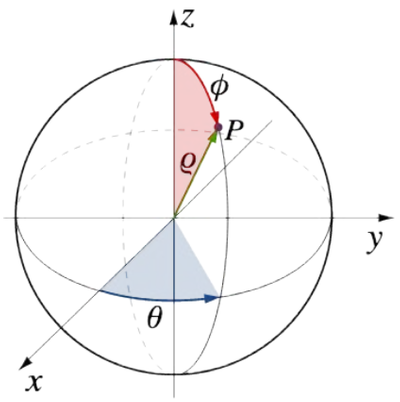

Figure 5.39 Spherical Coordinates
Spherical coordinates are useful when domains in R^3 have spherical symmetry.

Integrals in Spherical Coordinates

The Jacobian determinant of this transformation is the following:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix} = \rho^2 \sin\phi.$$ The (dx dy dz) differentials therefore are transformed to (ρ²sinφ dρ dθ dφ). Finally, you obtain the final integration formula:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

It's better to use this method in case of spherical domains and in case of functions that can be easily simplified, by the first fundamental relation of trigonometry, extended in R^3; in other cases it can be better to use cylindrical coordinates.

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Example

Integrate $f(x, y, z) = x^2 + y^2 + z^2$ over the domain $D = x^2 + y^2 + z^2 \leq 16$. In spherical coordinates, $f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) = \rho^2$, while the intervals of the transformed region T from D: $(0 \leq \rho \leq 4,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi)$. Therefore,

$$\iiint_D (x^2 + y^2 + z^2)\, dx\, dy\, dz = \iiint_T \rho^2 \ \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^4 \rho^4 d\rho \int_0^{2\pi} d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi \left[\frac{\rho^5}{5}\right]_0^4 d\phi$$

$$= 2\pi \left[\frac{\rho^5}{5}\right]_0^4 \left[-\cos\phi\right]_0^{\pi} = \frac{4096\pi}{5}.$$

Triple Integrals

For T ⊆ R^3, the triple integral over T is written as

$$\iiint_T f(x,y,z)\,dx\,dy\,dz.$$

KEY POINTS

- By convention, the triple integral has three integral signs (and a double integral has two integral signs); this is a notational convention which is convenient when computing a multiple integral as an iterated integral.
- If T is a domain that is normal with respect to the xy-plane and determined by the functions α (x,y) and β(x,y), then.
- To integrate a function with spherical symmetry such as $f(x,y,z) = x^2 + y^2 + z^2$, consider changing integration variable to spherical coordinates.

A triple integral is the integral of a function in three variables. For T ⊆ R3, the triple integral over T is written as $\iiint_T f(x,y,z)\,dx\,dy\,dz.$

Notice that, by convention, the triple integral has three integral signs (and a double integral has two integral signs); this is a notational convention which is convenient when computing a multiple integral as an iterated integral.

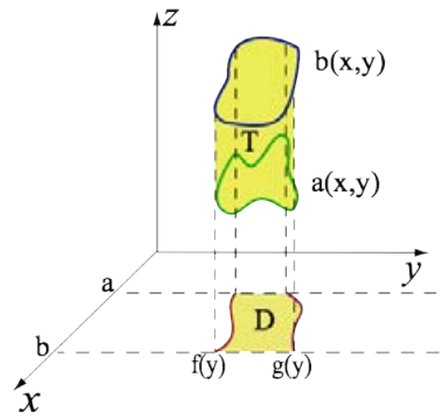
We have seen that double integrals can be evaluated over regions with a general shape. (See our atom on "Double integral over general region.") The extension of those formulae to triple integrals should be apparent. If T is a domain that is normal with respect to the xy-plane and determined by the functions α (x,y) and β(x,y), then $\iiint_T f(x,y,z)\,dx\,dy\,dz = \iint_D \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz\,dx\,dy$ (*Figure 5.40*).

Example 1

The volume of the parallelepiped of sides 4 × 6 × 5 may be obtained in two ways:

- By calculating the double integral of the function f(x, y) = 5 over the region D in the xy-plane which is the base of the

Figure 5.40 Graphical Representation of a Triple Integral



Example of domain in that is normal with respect to the xy-plane.

parallelepiped: $\iint_D 5 \; dx \, dy$

- By calculating the triple integral of the constant function 1 over the parallelepiped itself: $\iiint_{\text{parallelepiped}} 1 \, dx \, dy \, dz$

Example 2

Integrate $f(x, y, z) = x^2 + y^2 + z^2$ over the domain $D = \{x^2 + y^2 + z^2 \leq 16\}$.

Looking at the domain, it seems convenient to adopt the passage in spherical coordinates; in fact, the intervals of the variables that delimit the new T region are obviously:

$(0 \leq \rho \leq 4,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi)]\,.\, For the function, we get EQUATION[f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) = \rho^2$

Therefore,

$$\iiint_D (x^2 + y^2 + z^2) \, dx \, dy \, dz$$

$$\iiint_T \rho^2 \;\; \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin\phi \, d\phi \int_0^4 \rho^4 d\rho \int_0^{2\pi} d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi \left[\frac{\rho^5}{5}\right]_0^4 d\phi$$

$$= 2\pi \left[\frac{\rho^5}{5}\right]_0^4 \left[-\cos\phi\right]_0^{\pi} = \frac{4096\pi}{5}$$

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Change of Variables

One makes a change of variables to rewrite the integral in a more "comfortable" region, which can be described in simpler formulae.

KEY POINTS

- There exist three main "kinds" of changes of variable (to polar coordinate in R^2, and to cylindrical and spherical coordinates in R^3); however, more general substitutions can be made using the same principle.
- When changing integration variables, make sure that the integral domain also changes accordingly.
- Change of variable should be judiciously applied based on the built-in symmetry of the function to be integrated.

The limits of integration are often not easily interchangeable (without normality or with complex formulae to integrate). One makes a change of variables to rewrite the integral in a more "comfortable" region, which can be described in simpler formulae. To do so, the function must be adapted to the new coordinates.

For example, for the function $f(x, y) = (x - 1)^2 + \sqrt{y}$, if one adopts this substitution $x' = x - 1,\ y' = y$, therefore $x = x' + 1,\ y = y'$, one obtains the new function $f_2(x, y) = (x')^2 + \sqrt{y}$, which is simpler than the original form. When changing integration variables, however, make sure that the integral domain also changes accordingly. In the example, if integration is performed over x in [0,1] and y in [0,3], the new variables x' and y' vary over [-1,0] and [0,3], respectively. There exist three main "kinds" of changes of variable (one in R^2, two in R^3); however, more general substitutions can be made using the same principle.

1. Polar coordinates

The function to be integrated transforms as $f(x, y) \to f(\rho\cos\phi, \rho\sin\phi)$, and the integral accordingly changes as

$$\iint_D f(x, y)\, dx\, dy = \iint_T f(\rho\cos\phi, \rho\sin\phi)\rho\, d\rho\, d\phi.$$

2. Cylindrical coordinates

The function to be integrated transforms as $f(x, y, z) \to f(\rho\cos\phi, \rho\sin\phi, z)$ (*Figure 5.41*), and the integral accordingly changes as

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho\cos\phi, \rho\sin\phi, z)\rho\, d\rho\, d\phi\, dz.$$

3. Spherical coordinates

The function to be integrated transforms as $f(x, y, z) \longrightarrow f(\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$, and the integral accordingly changes as:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

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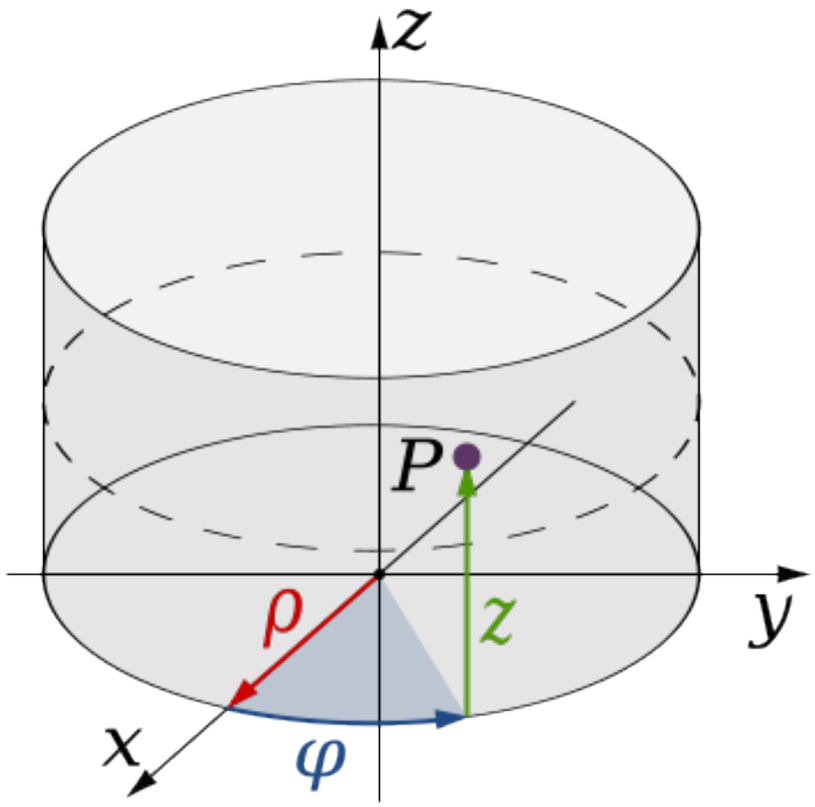
Figure 5.41 Cylindrical Coordinates



Changing to cylindrical coordinates may be useful depending on the setup of problem.

Applications of Multiple Integrals

Multiple integrals are used in many applications in physics and engineering.

KEY POINTS

- Given a set D $\subseteq R^n$ and an integrable function f over D, the average value of f over its domain is given by $\bar{f} = \frac{1}{m(D)} \int_D f(x)\,dx$, where m(D) is the measure of D.
- The gravitational potential associated with a mass distribution given by a mass measure dm on three-dimensional Euclidean space R^3 is $V(\mathbf{x}) = -\int_{\mathbf{R}^3} \frac{G}{|\mathbf{x}-\mathbf{r}|}\,dm(\mathbf{r})$.
- An electric field produced by a distribution of charges given by the volume charge density $\overrightarrow{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3} \rho(\vec{r}')\,d^3r'$.

As is the case with one variable, one can use the multiple integral to find the average of a function over a given set. Given a set D $\subseteq$ R^n and an integrable function f over D, the average value of f over its domain is given by $\bar{f} = \frac{1}{m(D)} \int_D f(x)\,dx$, where m(D) is the measure of D. Additionally, multiple integrals are used in many applications in physics and engineering. The examples below also show some variations in the notation.

Example 1

In mechanics, the moment of inertia is calculated as the volume integral (triple integral) of the density weighed with the square of the distance from the axis: $I_z = \iiint_V \rho r^2\,dV$.

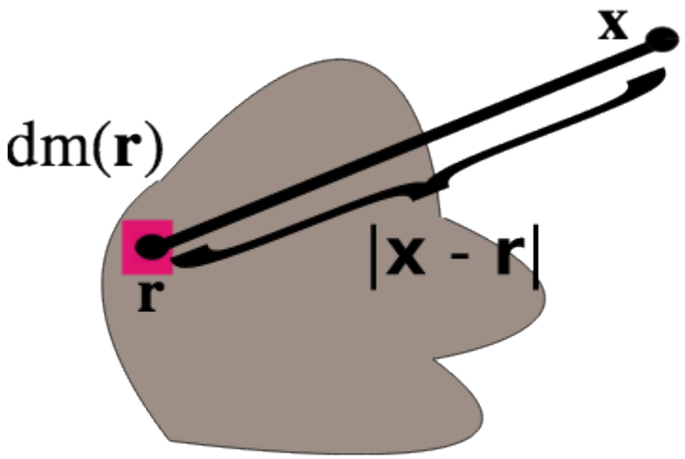
Example 2

The gravitational potential associated with a mass distribution given by a mass measure dm on three-dimensional Euclidean space R3 is $V(\mathbf{x}) = -\int_{\mathbf{R}^3} \frac{G}{|\mathbf{x}-\mathbf{r}|}\,dm(\mathbf{r})$ (*Figure 5.42*). If there is a continuous function ρ(x) representing the density of the distribution at x, so that dm(x) = ρ(x)d³x, where d³x is the Euclidean volume element, then the gravitational potential is $V(\mathbf{x}) = -\int_{\mathbf{R}^3} \frac{G}{|\mathbf{x}-\mathbf{r}|}\,\rho(\mathbf{r})\,d^3\mathbf{r}$.

Example 3

In electromagnetism, Maxwell's equations can be written using multiple integrals to calculate the total magnetic and electric fields. In the following example, the electric field produced by a

Figure 5.42 A Mass to be Integrated



Points x and r, with r contained in the distributed mass (gray) and differential mass dm(r) located at the point r.

distribution of charges given by the volume charge density $\rho(\vec{r})$ is obtained by a triple integral of a vector function: $\vec{E} = \frac{1}{4\pi\epsilon_0}\iiint \frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3}\rho(\vec{r}')\,d^3r'$. This can also be written as an integral with respect to a signed measure representing the charge distribution.

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Center of Mass and Inertia

The center of mass for a rigid body can be expressed as a triple integral.

KEY POINTS

- In the case of a single rigid body, the center of mass is fixed in relation to the body, and if the body has uniform density, it will be located at the centroid.
- In the case of a system of particles Pi, i = 1, ..., n , each with mass mi that are located in space with coordinates ri, i = 1, ..., n , the coordinates R of the center of mass is given as $\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$.
- If the mass distribution is continuous with the density ρ(r) within a volume V, the center of mass is expressed as $\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r})\mathbf{r}\,dV$.

Multiple integrals are used in many applications in physics and engineering. In this atom, we will see how center of mass can be calculated using multiple integrals.

The center of mass is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero. In the case of a single

rigid body, the center of mass is fixed in relation to the body, and if the body has uniform density, it will be located at the **centroid**. The center of mass may be located outside the physical body, as is sometimes the case for hollow or open-shaped objects, such as a horseshoe. In the case of a distribution of separate bodies, such as the planets of the Solar System, the center of mass may not correspond to the position of any individual member of the system (*Figure 5.43*).

A System of Particles

In the case of a system of particles P_i, i = 1, ..., n , each with mass m_i that are located in space with coordinates r_i, i = 1, ..., n , the coordinates R of the center of mass satisfy the condition $\sum_{i=1}^{n} m_i(\mathbf{r}_i - \mathbf{R}) = 0$. Solve this equation for R to obtain the formula, $\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i$ where M is the sum of the masses of all of the particles.

A Continuous Volume

If the mass distribution is continuous with the density ρ(r) within a volume V, then the integral of the weighted position coordinates of the points in this volume relative to the center of mass R is zero; that is,

$$\int_V \rho(\mathbf{r})(\mathbf{r} - \mathbf{R})dV = 0.$$

Solve this equation for the coordinates R to obtain

$$\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r})\mathbf{r}dV,$$

where M is the total mass in the volume. The integral is over the three dimensional volume, so it is a triple integral.

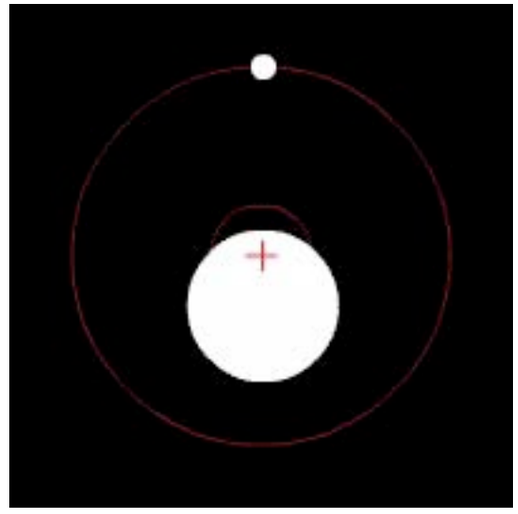

Figure 5.43 Center of Mass Two bodies orbiting around the center of mass inside one body

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Section 5

Vector Calculus

Vector Fields

Conservative Vector Fields

Line Integrals

Fundamental Theorem for Line Integrals

Green’s Theorem

Curl and Divergence

Parametric Surfaces and Surface Integrals

Surface Integrals of Vector Fields

Stokes' Theorem

The Divergence Theorem

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Vector Fields

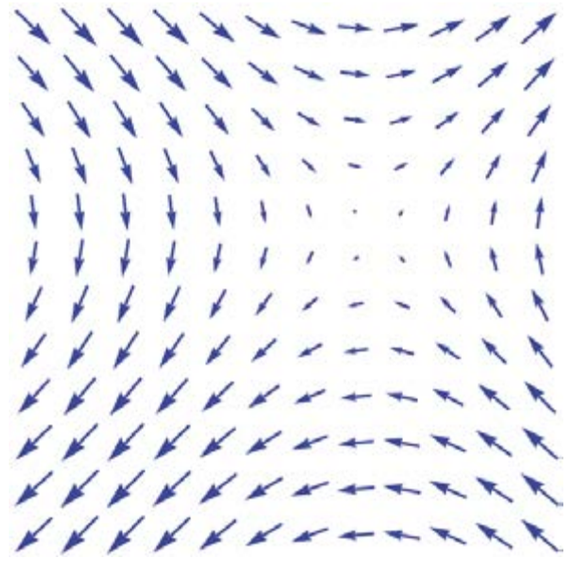
A vector field is an assignment of a vector to each point in a subset of Euclidean space.

KEY POINTS

- A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane.
- Vector fields can be constructed out of scalar fields using the gradient operator.
- Vector fields can be thought to represent the velocity of a moving flow in space, and this physical intuition leads to notions such as the divergence (the rate of change of volume of a flow) and curl (the rotation of a flow).

In vector calculus, a vector field is an assignment of a vector to each point in a subset of **Euclidean** space. A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane (*Figure 5.44*). Vector fields are often used to model the speed and direction of a moving fluid throughout space, for example, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.

Figure 5.44 Vector Field



A portion of the vector field given by $(V_x, V_y(siny, sinx))$

The elements of differential and integral calculus extend to vector fields in a natural way. When a vector field represents force, the line integral of a vector field represents the work done by a force moving along a path, and, under this interpretation, conservation of energy is exhibited as a special case of the fundamental theorem of calculus. Vector fields can be thought to represent the velocity of a moving flow in space, and this physical intuition leads to notions such as the divergence (the rate of change of volume of a flow) and curl (the rotation of a flow).

Gradient field

Vector fields can be constructed out of scalar fields using the **gradient** operator (denoted by the del: ∇). A vector field V defined on a set S is called a gradient field or a conservative field if there exists a real-valued function (a scalar field) f on S such that

$V = \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n} \right)$. The associated flow is called the gradient flow.

EXAMPLES

A vector field for the movement of air on Earth will associate for every point on the surface of the Earth a vector with the wind speed and direction for that point.

A gravitational field generated by any massive object is a vector field. For example, the gravitational field vectors for a spherically symmetric body would all point towards the sphere's center, with the magnitude of the vectors reducing as radial distance from the body increases.

Magnetic **field lines** can be revealed using small iron filings.

In the case of the velocity field of a moving fluid, a velocity vector is associated to each point in the fluid.

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Conservative Vector Fields

A conservative vector field is a vector field which is the gradient of a function, known in this context as a scalar potential.

KEY POINTS

- Conservative vector fields have the following property: The line integral from one point to another is independent of the choice of path connecting the two points; it is path-independent.
- Conservative vector fields are also irrotational, meaning that (in three dimensions) they have vanishing curl.
- A vector field v is said to be conservative if there exists a scalar field φ such that $\mathbf{v} = \nabla \varphi$

A conservative vector field is a vector field which is the **gradient** of a function, known in this context as a scalar potential. Conservative vector fields have the following property: The line integral from one point to another is independent of the choice of path connecting the two points; it is path-independent. Conversely, path independence is equivalent to the vector field's being conservative. Conservative vector fields are also irrotational, meaning that (in three dimensions) they have vanishing curl. In fact, an irrotational vector

field is necessarily conservative provided that a certain condition on the geometry of the domain holds: it must be simply connected.

Definition

A vector field v is said to be conservative if there exists a scalar field φ such that $\mathbf{v} = \nabla\varphi$. Here $\nabla\varphi$ denotes the gradient of φ. When the above equation holds, φ is called a scalar potential for $\mathbf{v}$.

For any scalar field φ : $\nabla \times \nabla\varphi = \mathbf{0}$. Therefore, the curl of a conservative vector field v is always 0. A vector field v, whose curl is zero, is called irrotational.

Path Independence

A key property of a conservative vector field is that its integral along a path depends only on the endpoints of that path, not the particular route taken. Suppose that $S \subseteq \mathrm{R}^3$ is a region of three-dimensional space, and that P is a rectifiable path in S with start point A and end point B. If $\mathbf{v} = \nabla\varphi$ is a conservative vector field, then the gradient theorem states that $\int_P \mathbf{v} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$. This holds as a consequence of the Chain Rule and the Fundamental Theorem of Calculus. An equivalent formulation of this is to say that $\oint \mathbf{v} \cdot d\mathbf{r} = 0$ for every closed loop in S.

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Line Integrals

A line integral is an integral where the function to be integrated is evaluated along a curve.

KEY POINTS

- The value of the line integral is the sum of the values of the field at all points on the curve, weighted by some scalar functionon the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve).
- Many simple formulae in physics (for example, W=F·s) have natural continuous analogs in terms of line integrals (W=$\int_C$ F· ds). The line integral finds the work done on an object moving through an electric or gravitational field, for example.
- In qualitative terms, a line integral in vector calculus can be thought of as a measure of the total effect of a given field along a given curve.

A line integral (sometimes called a path integral, contour integral, or curve integral) is an integral where the function to be integrated is evaluated along a curve.

The function to be integrated may be a scalar field or a **vector field**. The value of the line integral is the sum of the values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulae in physics (for example, W=F·s) have natural continuous analogs in terms of line integrals (W=$\int_C$ F· ds). The line integral finds the work done on an object moving through an electric or gravitational field, for example.

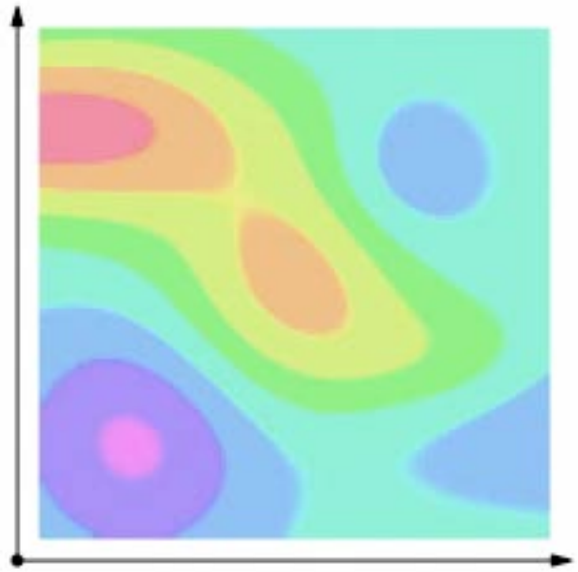
In qualitative terms, a line integral in vector calculus can be thought of as a measure of the total effect of a given field along a given curve. More specifically, the line integral over a scalar field can be interpreted as the area under the field carved out by a particular curve. This can be visualized as the surface created by z = f(x,y) and a curve C in the x-y plane. The line integral of f would be the area of the "curtain" created when the points of the surface that are directly over C are carved out (*Figure 5.45*).

Line Integral of a Scalar Field

For some scalar field f : U $\subseteq$ R^n $\rightarrow$ R, the line integral along a piecewise smooth curve C $\subset$ U is defined as

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt$$, where r: [a, b] $\rightarrow$ C is an arbitrary **bijective** parametrization of the curve C such that r(a) and r(b) give the endpoints of C and a.

Figure 5.45 Line Integral Over Scalar Field



The line integral over a scalar field f can be thought of as the area under the curve C along a surface z = f(x,y), described by the field.

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Line Integral of a Vector Field

For a vector field $F : U \subseteq R^n \to R^n$, the line integral along a piecewise smooth curve $C \subset U$, in the direction of r, is defined as $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$, where $\cdot$ is the dot product and r: [a, b] $\to$ C is a bijective parametrization of the curve C such that r(a) and r(b) give the endpoints of C.

Fundamental Theorem for Line Integrals

Gradient theorem says that a line integral through a gradient field can be evaluated from the field values at the endpoints of the curve.

KEY POINTS

- The gradient theorem implies that line integrals through irrotational vector fields are path-independent.
- Work done by conservative forces, described by a vector field, does not depend on the path followed by the object, but only the end points, as the above equation shows.
- Any conservative vector field can be expressed as the gradient of a scalar field.

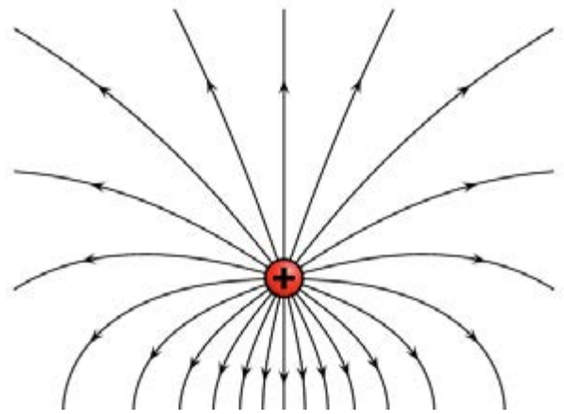
The gradient theorem, also known as the fundamental theorem of calculus for line integrals, says that a line integral through a gradient field can be evaluated by evaluating the original scalar field at the endpoints of the curve.

Let $\varphi : U \subseteq \mathrm{R}^n \to \mathrm{R}$. This holds that $\varphi\left(\mathbf{q}\right) - \varphi\left(\mathbf{p}\right) = \int_{\gamma[\mathbf{p},\,\mathbf{q}]} \nabla\varphi(\mathbf{r}) \cdot d\mathbf{r}$.

It is a generalization of the fundamental theorem of calculus to any curve in a plane or space (generally n-dimensional) rather than just the real line.

The gradient theorem implies that line integrals through irrotational **vector field**s are path-independent. In physics this theorem is one of the ways of defining a "**conservative force**" (*Figure 5.46*). By placing φ as potential, ∇φ is a conservative field. Work done by conservative forces does not depend on the path followed by the object, but only the end points, as the above equation shows.

Figure 5.46 Electric Field Lines of a Positive Charge



Electric field lines emanating from a point where positive electric charge is suspended over a negatively charged infinite sheet. Electric field is a vector field which can be represented as a gradient of a scalar field, called electric potential. Therefore, electric force is a conservative force.

The gradient theorem also has an interesting converse: any conservative vector field can be expressed as the gradient of a scalar field. Just like the gradient theorem itself, this converse has many striking consequences and applications in both pure and applied mathematics.

Proof

If φ is a differentiable function from some open subset U (of R^n) to R, and if r is a differentiable function from some closed interval [a,b] to U, then by the multivariate chain rule, the composite function φ ◦ r is differentiable on (a, b) and $\frac{d}{dt}(\varphi \circ \mathbf{r})(t) = \nabla\varphi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ for for all t in (a, b). Here the · denotes the usual inner product.

Now suppose the domain U of φ contains the differentiable curve γ with endpoints p and q (oriented in the direction from p to q). If r parametrizes γ for t in [a, b], then the above shows that

$$\int_\gamma \nabla\varphi(\mathbf{u}) \cdot d\mathbf{u}$$

$$= \int_a^b \nabla\varphi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

$$= \int_a^b \frac{d}{dt}\varphi(\mathbf{r}(t))dt = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)) = \varphi\left(\mathbf{q}\right) - \varphi\left(\mathbf{p}\right)$$

where the definition of the line integral is used in the first equality and the fundamental theorem of calculus is used in the third equality.

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Green's Theorem

Green's theorem gives relationship between a line integral around closed curve C and a double integral over plane region D bounded by C.

KEY POINTS

- Green's theorem is a special case of the Kelvin–Stokes theorem, when applied to a region in the xy-plane.
- Considering only two-dimensional vector fields, Green's theorem is equivalent to the two-dimensional version of the divergence theorem.
- Green's theorem can be used to compute area by line integral.

Green's theorem gives the relationship between a **line integral** around a simple closed curve C and a **double integral** over the plane region D bounded by C.

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$ where the path of integration along C is counterclockwise.

In physics, Green's theorem is mostly used to solve two-dimensional flow integrals, stating that the sum of fluid outflows at any point inside a volume is equal to the total outflow summed about an enclosing area. In plane geometry and area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.

Green's theorem is a special case of the Kelvin–Stokes theorem, when applied to a region in the xy-plane. Considering only two-dimensional vector fields, Green's theorem is equivalent to the two-dimensional version of the divergence theorem.

Green's theorem can be used to compute area by line integral, as shown in *Figure 5.47*. The area of D is given by:

$$A = \iint_D \mathrm{d}A.$$

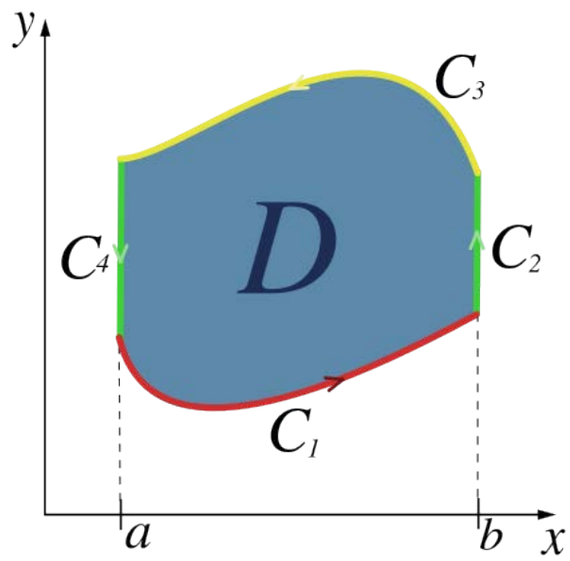

Figure 5.47 Computing area by line integral

D is a simple region with its boundary consisting of the curves C_1, C_2, C_3, C_4.

Provided we choose L and M such that: $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 1$. Then the area is given by:

$A = \oint_C x$. Possible formulas for the area of D include: $A = \oint_C x\,dy$, $A = -\oint_C y\,dx$, and $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$.

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Curl and Divergence

The four most important differential operators are gradient, curl, divergence, and Laplacian.

KEY POINTS

- The curl is a vector operator that describes the infinitesimal rotation of a three-dimensional vector field.
- The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.
- Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point in terms of a signed scalar.

Vector calculus studies various differential operators defined on scalar or vector fields, which are typically expressed in terms of the del operator (∇). The four most important differential operators are **gradient** (∇f), curl ($\nabla \times \mathbf{F}$), divergence ($\nabla \cdot \mathbf{F}$), and Laplacian ($\nabla^2 f = \nabla \cdot \nabla f$) (*Figure 5.48*).

Curl

The curl is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl

Figure 5.48 Four Most Important Differential Operators

Operation	Notation	Description	Domain/Range
Gradient	$\operatorname{grad}(f) = \nabla f$	Measures the rate and direction of change in a scalar field.	Maps scalar fields to vector fields.
Curl	$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$	Measures the tendency to rotate about a point in a vector field.	Maps vector fields to (pseudo)vector fields.
Divergence	$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$	Measures the magnitude of a source or sink at a given point in a vector field.	Maps vector fields to scalar fields.
Laplacian	$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$	A composition of the divergence and gradient operations.	Maps scalar fields to scalar fields.

Gradient, curl, divergence, and Laplacian are four most important differential operators.

of that field is represented by a vector. The attributes of this vector--length and direction-- characterize the rotation at that point.

The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation. If the vector field represents the flow velocity of a moving fluid, then the curl is the circulation density of the fluid. A vector field whose curl is zero is called irrotational. The curl is a form of differentiation for vector fields. The corresponding form of the fundamental theorem of calculus is Stokes' theorem, which relates the surface integral of the curl of a vector field to the line integral of the vector field around the boundary curve.

The curl of a vector field F, denoted by curl F or $\nabla \times$ F, is defined at a point in terms of its projection onto various lines through the point. If n is any unit vector, the projection of the curl of F onto n is defined to be the limiting value of a closed line integral in a plane orthogonal to n as the path used in the integral becomes infinitesimally close to the point, divided by the area enclosed.

Curl is defined by $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \ \stackrel{\text{def}}{=} \lim_{A \to 0} \left(\frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$.

Divergence

Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point in terms of a signed scalar. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

In physical terms, the divergence of a three dimensional vector field is the extent to which the vector field flow behaves like a source or a sink at a given point. It is a local measure of its "outgoingness"—the extent to which there is more exiting an infinitesimal region of space than entering it. If the divergence is nonzero at some point, then there must be a source or sink at that position.[1] (Note that we are imagining the vector field to be like the velocity vector field of a fluid (in motion) when we use the terms flow, sink, and so on.)

More rigorously, the divergence of a vector field F at a point p is defined as the limit of the net flow of F across the smooth boundary of a three-dimensional region V divided by the volume of V as V shrinks to p. Formally, $\nabla \cdot \mathbf{F}(p) = \lim_{V \to \{p\}} \iint_{S(V)} \frac{\mathbf{F} \cdot \mathbf{n}}{|V|} \, dS$, where |V | is the volume of V, S(V) is the boundary of V, and the integral is a

surface integral with n being the outward unit normal to that surface.

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Parametric Surfaces and Surface Integrals

A parametric surface is a surface in the Euclidean space R^3 which is defined by a parametric equation.

KEY POINTS

- Parametric representation is the most general way to specify a surface. Thecurvature and arc length of curves on the surface can both be computed from a given parametrization.
- The same surface admits many different parametrizations.
- A surface integral is a definite integral taken over a surface. It can be thought of as the double integral analog of the line integral.

Parametric Surface

A parametric surface is a surface in the **Euclidean** space R^3 which is defined by a parametric equation with two parameters: $\vec{r} : \mathrm{R}^2 \to \mathrm{R}^3$. Parametric representation is the most general way to specify a surface. Surfaces that occur in two of the main theorems of vector calculus, **Stokes' theorem** and the divergence theorem, are frequently given in a parametric form. The curvature and arc length

of curves on the surface can both be computed from a given parametrization.

Examples

- The simplest type of parametric surfaces is given by the graphs of functions of two variables.
- Using the spherical coordinates, the unit sphere can be parameterized by
  $\vec{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi.$
- The straight circular cylinder of radius R about x-axis has the following parametric representation:
  $\vec{r}(x, \phi) = (x, R\cos\phi, R\sin\phi).$

The same surface admits many different parametrizations. For example, the coordinate z-plane can be parametrized as $\vec{r}(u, v) = (au + bv, cu + dv, 0)$ for any constants a, b, c, d such that ad − bc ≠ 0, i.e. the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is **invertible**.

Surface integral

A surface integral is a definite integral taken over a surface (*Figure 5.49*). It can be thought of as the double integral analog of the line integral. Given a surface, one may integrate over its scalar fields (that is, functions which return scalars as values), and vector fields (that is, functions which return vectors as values). Surface integrals have many applications in physics, particularly within the classical theory of electromagnetism. We will study surface integral of vector fields and related theorems in the following atoms.

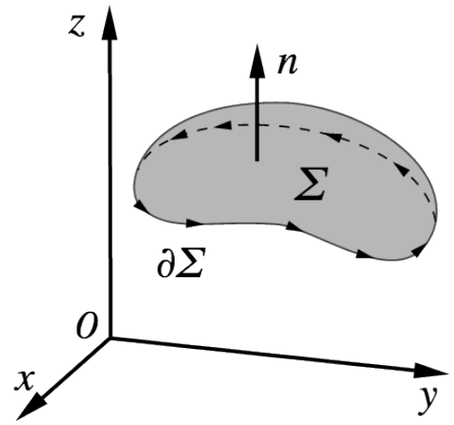

Figure 5.49 Kelvin-Stokes' Theorem
An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary ∂Σ, and the "normal" vector n.

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Surface Integrals of Vector Fields

The surface integral of vector fields can be defined component-wise according to the definition of the surface integral of a scalar field.

KEY POINTS

- The flux is defined as the quantity of fluidflowing through S in unit amount of time.
- To find the flux, we need to take the dot product of v with the unit surface normalto S at each point, which will give us ascalar field, and integrate the obtained field.
- This is expressed as

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_S \mathbf{v} \cdot \mathbf{n}\, dS = \iint_T \mathbf{v}(\mathbf{x}(s,t)) \cdot \left(\frac{\partial \mathbf{x}}{\partial s} \times \frac{\partial \mathbf{x}}{\partial t} \right) ds\, dt.$$

Consider a vector field v on S; that is, for each x in S, v(x) is a vector. The surface integral of vector fields can be defined component-wise according to the definition of the surface integral of a **scalar field**; the result is a vector. This applies, for example, in the expression of the electric field at some fixed point due to an electrically charged surface, or the gravity at some fixed point due to a sheet of material.

Alternatively, if we integrate the normal component of the vector field, the result is a scalar. Imagine that we have a fluid flowing through S, such that v(x) determines the velocity of the fluid at x. The **flux** is defined as the quantity of fluid flowing through S in unit amount of time.

This illustration implies that if the vector field is tangent to S at each point, then the flux is zero, because the fluid just flows in parallel to S, and neither in nor out. This also implies that if v does not just flow along S-- that is, if v has both a tangential and a normal component-- then only the normal component contributes to the flux. Based on this reasoning, to find the flux, we need to take the dot product of v with the unit surface normal to S at each point, which will give us a scalar field, and integrate the obtained field as above. We find the formula

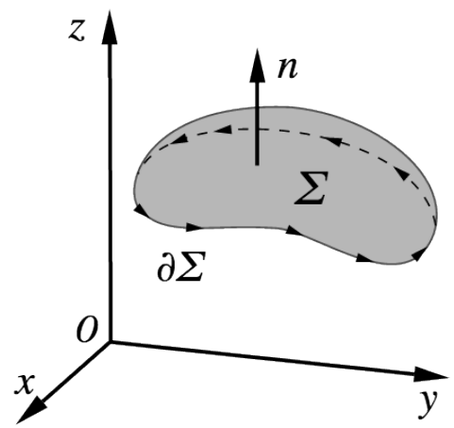

Figure 5.50
Kelvin-Stokes' Theorem
An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary ∂Σ, and the "normal" vector n.

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_S \mathbf{v} \cdot \mathbf{n}\, dS = \iint_T \mathbf{v}(\mathbf{x}(s,t)) \cdot \left(\frac{\partial \mathbf{x}}{\partial s} \times \frac{\partial \mathbf{x}}{\partial t} \right) ds\, dt$$ (*Figure 5.50*).

The cross product on the right-hand side of this expression is a surface normal determined by the parametrization. This formula defines the integral on the left (note the dot and the vector notation for the surface element).

EXAMPLE

An electric field from a point charge (Q) is given as $\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}$, where r is the position vector and $\hat{r}$ is a unit vector in radial direction. If the charge is located at the center of a sphere with a radius R, the surface integral of the electric field over the surface is calculated at the following:

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}\int_S \frac{1}{|\mathbf{r}|^2} dS = \frac{Q}{\varepsilon_0}.$$

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Stokes' Theorem

Stokes' theorem relates the integral of the curl of a vector field over a surface to the line integral of the field around the boundary.

KEY POINTS

- The generalized Stokes' theorem says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative dω over the whole of Ω.
- Given a vector field, the Kelvin-Stokes theorem relates the integral of the curl of the vector field over some surface to the line integral of the vector field around the boundary of the surface. The Kelvin–Stokes theorem is a special case of the generalized Stokes' theorem.
- By applying the Stokes' theorem, you can show that the work done by electric field is path-independent.

The generalized Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus. Stokes' theorem says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative dω over the whole of Ω, i.e. $\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$.

The Kelvin–Stokes theorem, also known as the **curl** theorem, is a theorem in vector calculus on R^3 (*Figure 5.51*). Given a vector field, the theorem relates the integral of the curl of the vector field over some surface to the line integral of the vector field around the boundary of the surface. The Kelvin–Stokes theorem is a special case of the "generalized Stokes' theorem." If F is a smooth vector field on R^3, then $\oint_\Gamma \mathbf{F}\, d\Gamma = \iint_S \nabla \times \mathbf{F}\, dS$ (1).

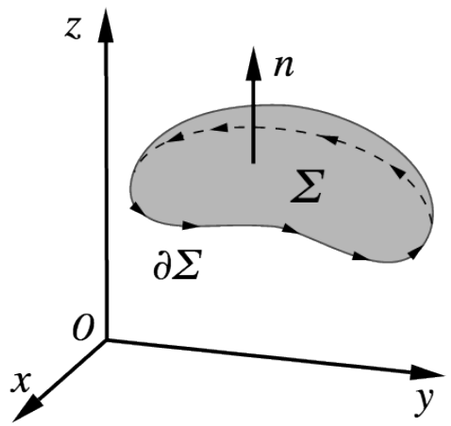

Figure 5.51
Kelvin-Stokes' Theorem
An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary ∂Σ, and the "normal" vector n.

Application in Electromagnetism

Electric field is a conservative vector field. Therefore, electric field can be written as a gradient of a scalar field: $\mathbf{E} = -\nabla\varphi$. Applying the Kelving-Stokes theorem and substituting $\mathbf{F} = \mathbf{E} = -\nabla\varphi$ in (1), we get $\oint_\Gamma \mathbf{E}\, d\Gamma = -\iint_S \nabla \times (\nabla\varphi)\, dS$. Since $\nabla \times \nabla f = 0$ for an arbitrary function f, we derive $\oint_\Gamma \mathbf{E}\, d\Gamma = 0$. As we have seen in our previous atom on gradient theorem, this simply means $\int_P \mathbf{E} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$, which is equivalent to saying that work done by the electric field only depends on the initial and final point of the motion. The scalar field φ in the case of electromagnetism is called **electric potential**.

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The Divergence Theorem

The divergence theorem relates the flow of a vector field through a surface to the behavior of the vector field inside the surface.

KEY POINTS

- The divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volumeintegral of the divergence over the region inside the surface.
- In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions.
- Applying the divergence theorem, we can check that the equation $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ is nothing but an equation describing Coulomb force written in a differential form.

The **divergence** theorem, also known as Gauss's theorem or Ostrogradsky's theorem, relates the flow (that is, **flux**) of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region. The divergence theorem is an important result for the mathematics of engineering, in particular for electrostatics and fluid dynamics.

In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions. In one dimension, it is equivalent to the fundamental theorem of calculus. The theorem is a special case of the generalized Stokes' theorem.

Theorem

Suppose V is a subset of R^n (in the case of n = 3, V represents a volume in 3D space) which is compact and has a piecewise smooth boundary S (also indicated with ∂V=S). If F is a continuously differentiable vector field defined on a neighborhood of V, then we have $\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oint_S (\mathbf{F} \cdot \mathbf{n})\, dS$. The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V (*Figure 5.52*).

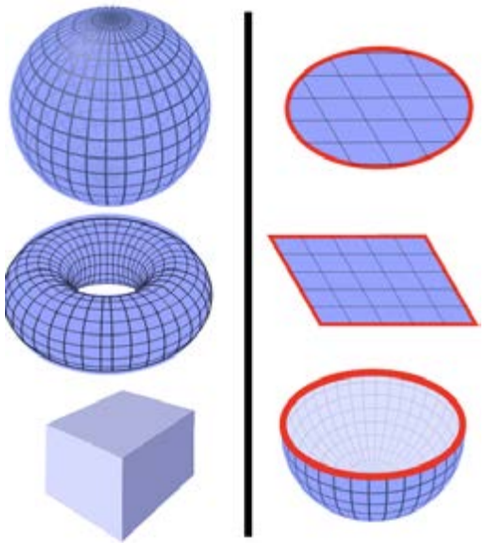

Figure 5.52 The Divergence Theorem

The divergence theorem can be used to calculate a flux through a closed surface that fully encloses a volume, like any of the surfaces on the left. It can not directly be used to calculate the flux through surfaces with boundaries, like those on the right. (Surfaces are blue, boundaries are red.)

Example

The first equation of the Maxwell's equations is often written as $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ in a differential form, where ρ is the electric density. Let's consider a system with a point charge Q located at the origin. We will apply the divergence theorem for a sphere of radius R, whose center is also at the origin. Substituting E for F in the relationship of the divergence theorem, the left hand side (LHS) becomes $\iiint_V (\nabla \cdot \mathbf{E})\, dV = \iiint_V (\frac{\rho}{\varepsilon_0}) dV = \frac{Q}{\varepsilon_0}$. The surface integral on the right hand side (RHS) becomes $\int_S (\mathbf{E} \cdot \mathbf{n})\, dS = 4\pi R^2 E$. Combining RHS and LHS, we get $E = \frac{Q}{4\pi\varepsilon_0 R^2}$. This is nothing but the electric field for the Coulomb force.

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Section 6

Second-Order Linear Equations

Second-Order Linear Equations

Nonhomogeneous Linear Equations

Applications of Second-Order Differential Equations

Series Solutions

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Second-Order Linear Equations

A second-order linear differential equation has the form $\frac{d^2y}{dt^2} + A_1(t)\frac{dy}{dt} + A_2(t)y = f(t)$, where A_1(t), A_2(t), and f(t) are continuous functions.

KEY POINTS

- Linear differential equations are of the form $L[y(t)] = f(t)$, where $L_n(y) \equiv \frac{d^n y}{dt^n} + A_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + A_{n-1}(t)\frac{dy}{dt} + A_n(t)y$.
- When f(t)=0, the equations are called homogeneous linear differential equations. (Otherwise, the equations are called nonhomogeneous equations).
- Linear differential equations are differential equations that have solutions which can be added together to form other solutions.

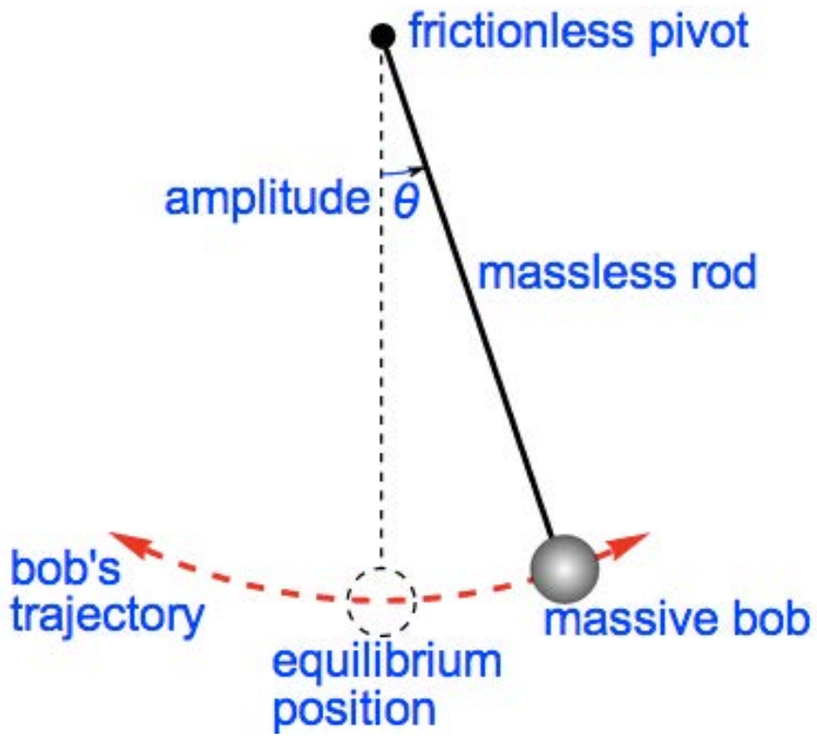
Linear **differential equations** are of the form $Ly = f$, where the differential operator L is a linear operator, y is the unknown function (such as a function of time y(t)), and the right hand side f is a given function of the same nature as y (called the source term). For a function dependent on time, we may write the equation more expressly as $Ly(t) = f(t)$ and, even more precisely, by bracketing $L[y(t)] = f(t)$.

The linear operator L may be considered to be of the form $L_n(y) \equiv \frac{d^n y}{dt^n} + A_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + A_{n-1}(t)\frac{dy}{dt} + A_n(t)y$. The linearity condition on L rules out operations such as taking the square of the derivative of y, but permits, for example, taking the second derivative of y. It is convenient to rewrite this equation in an operator form, $L_n(y) \equiv \left[D^n + A_1(t)D^{n-1} + \cdots + A_{n-1}(t)D + A_n(t)\right] y$, where D is the differential operator d/dt (i.e. Dy = y', $D^2y = y''$,...) and the A_n are given functions. Such an equation is said to have order n, the index of the highest derivative of y that is involved.

Second-Order Linear Differential Equations

A second-order linear differential equation has the form $\frac{d^2y}{dt^2} + A_1(t)\frac{dy}{dt} + A_2(t)y = f(t)(1)$, where A_1(t), A_2(t), and f(t) are continuous functions (*Figure 5.53*). When f(t)=0, the equations are called homogeneous second-order **linear** differential equations. (Otherwise, the equations are called nonhomogeneous equations).

Figure 5.53 Simple Pendulum



A simple pendulum, under the conditions of no damping and small amplitude, is described by a equation of motion which is a second-order linear differential equation.

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Nonhomogeneous Linear Equations

Nonhomogeneous second-order linear equation are of the the form: $\frac{d^2y}{dt^2} + A_1(t)\frac{dy}{dt} + A_2(t)y = f(t)$, where f(t) is nonzero.

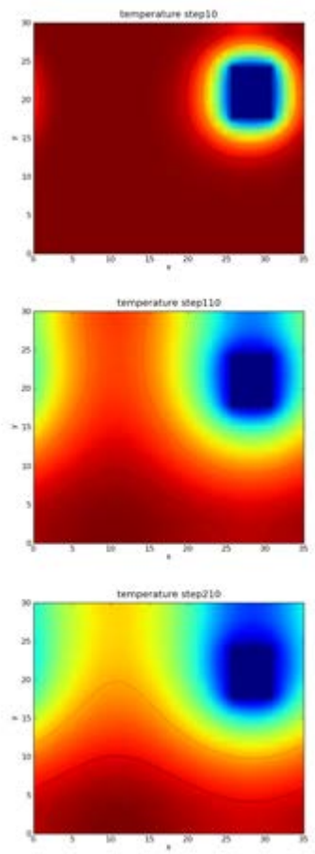
KEY POINTS

- Examples of homogeneous or nonhomogeneous second-order linear differential equation can be found in many different disciplines such as physics, economics, and engineering.
- In simple cases, for example, where the coefficients $A_1(t)$ and $A_2(t)$ are constants, the equation can be analytically solved. In general, the solution of the differential equation can only be obtained numerically.
- Linear differential equations are differential equations that have solutions which can be added together to form other solutions.

In the previous atom, we learned that a second-order linear differential equation has the form $\frac{d^2y}{dt^2} + A_1(t)\frac{dy}{dt} + A_2(t)y = f(t)$ (1), where $A_1(t)$, $A_2(t)$, and f(t) are continuous functions. When f(t)=0, the equations are called homogeneous second-order linear

differential equations. Otherwise, the equations are called nonhomogeneous equations. Examples of homogeneous or nonhomogeneous second-order linear differential equation can be found in many different disciplines such as physics, economics, and engineering (*Figure 5.54*).

Figure 5.54 Heat Transfer



Phenomena, such as heat transfer, can be described using nonhomogeneous second-order linear differential equations.

In simple cases, for example, where the coefficients $A_1(t)$ and $A_2(t)$ are constants, the equation can be analytically solved. (Either the method of undetermined coefficients or the method of variation of parameters can be adopted.) In general, the solution of the differential equation can only be obtained numerically. However, there is a very important property of the linear differential equation, which can be useful in finding solutions.

Linearity

Linear differential equations are differential equations that have solutions which can be added together to form other solutions. If $y_1(t)$ and $y_2(t)$ are both solutions of the second-order linear differential equations given in (1), then any arbitrary linear combination of $y_1(t)$ and $y_2(t)$, that is $y(x) = c_1 y_1(t) + c_2\, y_2(t)$ for constants c_1 and c_2, is also a solution of the differential equation (1). This can be confirmed by substituting $y(x) = c_1 y_1(t) + c_2\, y_2(t)$ in the Eq.(1) and using the fact that both $y_1(t)$ and $y_2(t)$ are solutions of Eq. (1).

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Applications of Second-Order Differential Equations

A second-order linear differential equation can be commonly found in physics, economics, and engineering.

KEY POINTS

- An ideal spring with a spring constant k is described by the simple harmonic oscillation, whose equation of motion is given in the form of a homogeneous second-order linear differential equation: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0$.
- Adding the damping term in the equation of motion, the equation of motion is given as.
- Adding the external force term to the damped harmonic oscillator, we get an nonhomogeneous second-order linear differential equation : $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2x = \frac{F(t)}{m}$.

Examples of homogeneous or nonhomogeneous second-order linear differential equation can be found in many different disciplines, such as physics, economics, and engineering. In this atom, we will learn about the harmonic oscillator, which is one of the simplest yet most important mechanical system in physics.

Harmonic oscillator

In classical mechanics, a **harmonic oscillator** is a system that, when displaced from its equilibrium position, experiences a restoring force, F, proportional to the displacement, x: $\overrightarrow{F} = -k\overrightarrow{x}$, where k is a positive constant. The system under consideration could be an object attached to a spring, a pendulum, etc. Electronic circuits such as RLC circuits are also described by similar equations.

Simple harmonic oscillation: If F is the only force acting on the system, the system is called a simple harmonic oscillator, and it undergoes simple harmonic motion: sinusoidal oscillations about the equilibrium point, with a constant amplitude and a constant frequency. The equation of motion is given as $F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$. Therefore, we end up with a homogeneous second-order linear differential equation $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0$. Note that the function $x(t) = A\cos\left(\omega_0 t + \phi\right)$ satisfies the equation where $\omega_0 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$. ω_0 is called angular velocity, and the constants (A, ϕ) are determined from initial conditions of the motion.

Damped harmonic oscillator: In real oscillators, friction (or damping) slows the motion of the system. In many vibrating systems the frictional force F_f can be modeled as being proportional

to the velocity v of the object: $F_f = -cv$, where c is called the viscous damping coefficient. Including this additional term, the equation of motion is given as $\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$, where $\zeta = \frac{c}{2\sqrt{mk}}$ is called the "damping ratio" (*Figure 5.55*).

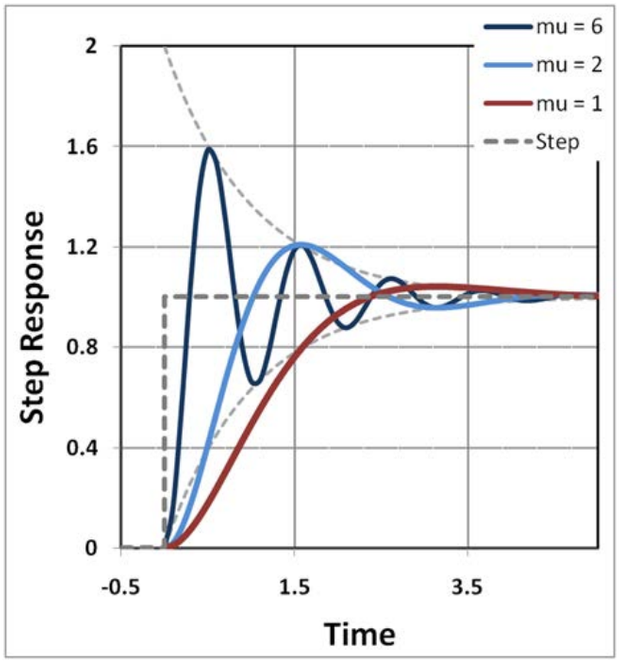

Figure 5.55
Damped Harmonic Oscillators
A solution of damped harmonic oscillator. Curves in different colors show various responses depending on the damping ratio.

Driven harmonic oscillator: Driven harmonic oscillators are damped oscillators further affected by an externally applied force F(t). Newton's 2nd law (F=ma) takes the form $F(t) - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$. It is usually rewritten into the form $\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m}$, which is a nonhomogeneous second-order linear differential equation.

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Series Solutions

The power series method is used to seek a power series solution to certain differential equations.

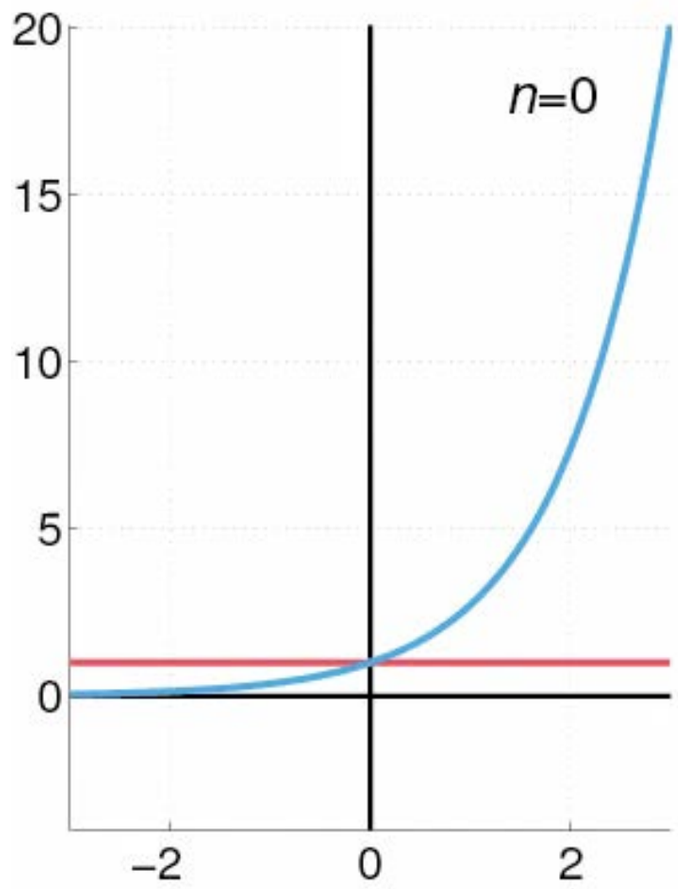
KEY POINTS

- The power series method calls for the construction of a power series solution $f'' + \frac{a_1(z)}{a_2(z)} f' + \frac{a_0(z)}{a_2(z)} f = 0.$
- The method assumes a power series with unknown coefficients, then substitutes that solution into the differential equation to find a recurrence relation for the coefficients.
- Hermit differential equation:

$$f = A_0 \left(1 + \frac{-1}{2}x^2 + \frac{-1}{8}x^4 + \frac{-7}{240}x^6 + \cdots\right) + A_1 \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)$$

The power series method is used to seek a power series solution to certain differential equations (*Figure 5.56*). In general, such a solution assumes a power series with unknown coefficients, then substitutes that solution into the differential equation to find a **recurrence relation** for the coefficients.

Figure 5.56 Maclaurin Power Series of an Exponential Function



The exponential function (in blue), and the sum of the first n+1 terms of its Maclaurin power series (in red). Using power series, a linear differential equation of a general form may be solved.

Method

Consider the second-order linear differential equation $a_2(z)f''(z) + a_1(z)f'(z) + a_0(z)f(z) = 0$. Suppose a_2 is nonzero for all z. Then we can divide throughout to obtain $f'' + \frac{a_1(z)}{a_2(z)} f' + \frac{a_0(z)}{a_2(z)} f = 0.$

Suppose further that a_1/a_2 and a_0/a_2 are analytic functions. The

power series method calls for the construction of a power series solution $f = \sum_{k=0}^{\infty} A_k z^k$. After substituting the power series form, recurrence relations for A_k is obtained, which can be used to reconstruct f.

Example

Let us look at the case known as Hermit differential equation: $f'' - 2zf' + \lambda f = 0;\ \lambda = 1$. We can try to construct a series solution

$$f = \sum_{k=0}^{\infty} A_k z^k$$

$$f' = \sum_{k=0}^{\infty} k A_k z^{k-1}$$

$$f'' = \sum_{k=0}^{\infty} k(k-1) A_k z^{k-2}$$

substituting these in the differential equation

$$\sum_{k=0}^{\infty} k(k-1) A_k z^{k-2} - 2z \sum_{k=0}^{\infty} k A_k z^{k-1} + \sum_{k=0}^{\infty} A_k z^k = 0$$

$$= \sum_{k=0}^{\infty} k(k-1) A_k z^{k-2} - \sum_{k=0}^{\infty} 2k A_k z^k + \sum_{k=0}^{\infty} A_k z^k$$

making a shift on the first sum

$$= \sum_{k+2=0}^{\infty} (k+2)((k+2)-1) A_{k+2} z^{(k+2)-2} - \sum_{k=0}^{\infty} 2k A_k z^k + \sum_{k=0}^{\infty} A_k z^k$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) A_{k+2} z^k - \sum_{k=0}^{\infty} 2k A_k z^k + \sum_{k=0}^{\infty} A_k z^k$$

$$= \sum_{k=0}^{\infty} \left((k+2)(k+1) A_{k+2} + (-2k+1) A_k \right) z^k.$$

If this series is a solution, then all these coefficients must be zero, so: $(k+2)(k+1)A_{k+2} + (-2k+1)A_k = 0$. We can rearrange this to get a recurrence relation for A_{k+2}:

$$A_{k+2} = \frac{(2k-1)}{(k+2)(k+1)} A_k.$$

Now, we have $A_{k+2} = \frac{(2k-1)}{(k+2)(k+1)} A_k$, and all coefficients with larger indices can be similarly obtained using the recurrence relation. The series solution is:

$$f = A_0 \left(1 + \frac{-1}{2} x^2 + \frac{-1}{8} x^4 + \frac{-7}{240} x^6 + \cdots \right) + A_1 \left(x + \frac{1}{6} x^3 + \frac{1}{24} x^5 + \frac{1}{112} x^7 + \cdots \right)$$

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Algebraic

containing only numbers, letters, and arithmetic operators

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Analytic function

a real valued function which is uniquely defined through its derivatives at one point

---

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Chapter 4 - Infinite Sequences and Series

Antiderivative

an indefinite integral

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Area

a measure of the extent of a surface measured in square units

---

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Arithmetic mean

the measure of central tendency of a set of values, computed by dividing the sum of the values by their number; commonly called the mean or the average

---

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Chapter 2 - Applications of Integration

Asymptote

a straight line which a curve approaches arbitrarily closely, as they go to infinity

---

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Chapter 1 - Limits

Average

any measure of central tendency, especially any mean, median, or mode

---

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Axis

a fixed, one-dimensional figure, such as a line or arc, with an origin and orientation and such that its points are in one-to-one correspondence with a set of numbers; an axis forms part of the basis of a space or is used to position and locate data in a graph (a coordinate axis)

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Azimuth

an arc of the horizon intercepted between the meridian of the place and a vertical circle passing through the center of any object

---

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Bicontinuous

homomorphic or of structure-preserving mapping

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Bijective

both injective and surjective

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Binary

the bijective base-2 numeral system, which uses only the digits 0 and 1

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Cartesian

of or pertaining to co-ordinates based on mutually orthogonal axes

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Centroid

the point at the center of any shape, sometimes called the center of area or the center of volume

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Chapter 5 - Multiple Integrals

Chain rule

formula for computing the derivative of the composition of two or more functions

---

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Commutative

such that the order in which the operands are taken does not affect their image under the operation

---

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Completeness of the real numbers

completeness implies that there are not any “gaps” or “missing points” in the real number line

---

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Chapter 1 - Limits

Complex analysis

theory of functions of a complex variable; a branch of mathematical analysis that investigates functions of complex numbers

---

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Composite

a function of a function

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Concave

curved like the inner surface of a sphere or bowl

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Chapter 2 - Applications of Differentiation

Conditional convergence

A series or integral is said to be conditionally convergent if it converges but does not converge absolutely.

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Cone

a surface of revolution formed by rotating a segment of a line around another line that intersects the first line

---

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Conservative force

a force with the property that the work done in moving a particle between two points is independent of the path taken

---

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Continuity

lack of interruption or disconnection; the quality of being continuous in space or time

---

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Continuous function

a function whose value changes only slightly when its input changes slightly

---

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Convergence

of a sequence, to have a limit

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Convergence tests

methods of testing for the convergence, conditional convergence, absolute convergence, interval of convergence, or divergence of an infinite series

---

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Convex

curved or bowed outward like the outside of a bowl or sphere or circle

---

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Chapter 2 - Applications of Differentiation

Coordinate system

a number representing the position of a point along a line, arc, or similar one-dimensional figure

---

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Coordinates

a number representing the position of a point along a line, arc, or similar one-dimensional figure

---

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Critical point

a maximum, minimum, or point of inflection on a curve; a point at which the derivative of a function is zero or undefined

---

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Cross product

also called a vector product; results in a vector which is perpendicular to both of the vectors being multiplied and therefore normal to the plane containing them

---

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Cuboid

a parallelepiped having six rectangular faces

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Chapter 5 - Partial Derivatives

Curl

the vector field denoting the rotationality of a given vector field

---

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Curvature

the degree to which an objet deviates from being flat

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Curve

a simple figure containing no straight portions and no angles

---

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Cylinder

a surface created by projecting a closed two-dimensional curve along an axis intersecting the plane of the curve

---

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Cylindrical coordinates

a three-dimensional coordinate system that specifies point positions by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis

---

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Decay

To change by undergoing fission, by emitting radiation, or by capturing or losing one or more electrons.

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Definite integral

the integral of a function between an upper and lower limit

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Definite integrals

the integral of a function between an upper and lower limit

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Derivative

A measure of how a function changes as its input changes

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Deterministic

having exactly predictable time evolution

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Chapter 5 - Partial Derivatives

Difference quotient

the function difference ΔF divided by the point difference Δx: ΔF(x)/Δx

---

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Differentiable

having a derivative, said of a function whose domain and co-domain are manifolds

---

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Differential equations

an equation involving the derivatives of a function

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Differential geometry

the study of geometry using differential calculus

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Differentiation

a vector quantity which denotes distance with a directional component

---

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Displacement

a vector quantity which denotes distance with a directional component

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Divergence

the set of all possible mathematical entities (points) where a given function is defined

---

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Domain

the set of all possible mathematical entities (points) where a given function is defined

---

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Double integral

An integral extended to functions of more than one real variable

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E

the base of the natural logarithm, 2.718281828459045...

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Eccentricity

the ratio-- constant for any particular conic section-- of the distance of a point from the focus to its distance from the directrix

---

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Chapter 5 - Vector Functions

Electric potential

the potential energy per unit charge at a point in a static electric field; voltage

---

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Ellipse

a closed curve; the locus of a point such that the sum of the distances from that point to two other fixed points (called the foci of the ellipse) is constant; equivalently, the conic section that is the intersection of a cone with a plane that does not intersect the base of the cone

---

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Error

the difference between a measured or calculated value and a true one

---

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Euclidean

adhering to the principles of traditional geometry, in which parallel lines are equidistant

---

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Chapter 5 - Vector Calculus

Euclidean space

ordinary two- or three-dimensional space (and higher dimensional generalizations), characterized by an infinite extent along each dimension and a constant distance between any pair of parallel lines

---

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Exponent

the power to which a number, symbol or expression is to be raised:f or example, the 3 in x3.

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Exponential

any function that has an exponent as an independent variable

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Chapter 3 - Inverse Functions: Exponential, Logarithmic, and Trigonometric Functions

Exponential function

any function in which an independent variable is in the form of an exponent; they are the inverse functions of logarithms

---

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Exponential growth

The growth in the value of a quantity, in which the rate of growth is proportional to the instantaneous value of the quantity; for example, when the value has doubled, the rate of increase will also have doubled. The rate may be positive or negative.

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Field lines

a line of constant strength in a field

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Flux

the rate of transfer of energy (or another physical quantity) through a given surface, specifically electric flux, magnetic flux

---

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Force

a physical quantity that denotes ability to push, pull, twist or accelerate a body which is measured in a unit dimensioned in mass × distance/time² (ML/T^2): SI: newton (N); CGS: dyne (dyn)

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Fubini's theorem

a result which gives conditions under which it is possible to compute a double integral using iterated integrals

---

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Function

A relation in which each element of the domain is associated with exactly one element of the co-domain

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Gradients

of a function y = f(x) or the graph of such a function, the rate of change of y with respect to x; that is, the amount by which y changes for a certain (often unit) change in x

---

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Graph

A diagram displaying data; in particular one showing the relationship between two or more quantities, measurements or indicative numbers that may or may not have a specific mathematical formula relating them to each other.

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Chapter 2 - Applications of Differentiation

Gravitational constant

an empirical physical constant involved in the calculation of the gravitational attraction between objects with mass

---

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Gross domestic product

a measure of the economic production of a particular territory in financial capital terms over a specific time period

---

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Harmonic oscillator

a system which, when displaced from its equilibrium position, experiences a restoring force proportional to the displacement according to Hooke's law, where k is a positive constant

---

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Hyperbola

a conic section formed by the intersection of a cone with a plane that intersects the base of the cone and is not tangent to the cone

---

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Hypersurface

a n-dimensional surface in a space (often a Euclidean space) of dimension n+1

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Hypervolumes

a volume in more than three dimensions

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Hypotenuse

the side of a right triangle opposite the right angle

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Implicit

implied indirectly, without being directly expressed

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Chapter 2 - Derivatives

Improper integral

an integral where at least one of the endpoints is taken as a limit, either to a specific number or to infinity

---

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Indeterminate

not accurately determined or determinable

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Infinitesimal

a non-zero quantity whose magnitude is smaller than any positive number

---

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Infinity

a number that has an infinite numerical value that cannot be counted

---

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Injective

of, relating to, or being an injection: such that each element of the image (or range) is associated with at most one element of the preimage (or domain); inverse-deterministic

---

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Integral

also sometimes called antiderivative; the limit of the sums computed in a process in which the domain of a function is divided into small subsets and a possibly nominal value of the function on each subset is multiplied by the measure of that subset, all these products then being summed

---

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Chapter 2 - Integrals

Integral tests

a method used to test infinite series of non-negative terms for convergence by comparing it to improper integrals

---

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Integrand

the function that is to be integrated

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Integration

the operation of finding the region in the x-y plane bound by the function

---

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Inverse

a function that undoes another function

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Chapter 3 - Inverse Functions: Exponential, Logarithmic, and Trigonometric Functions

Inverse function

a function that does exactly the opposite of another

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Invertible

capable of being inverted; having an inverse

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Irrational

of a real number, that cannot be written as the ratio of two integers

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Irreducible

unable to be factorized into polynomials of lower degree, as (x^2 + 1)

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Limits

a value to which a sequence or function converges

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Line integral

An integral the domain of whose integrand is a curve.

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Linear

having the form of a line; straight

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Linear equation

a polynomial equation of the first degree (such as x = 2y - 7)

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Linearity

a relationship between several quantities which can be considered proportional and expressed in terms of linear algebra; any mathematical property of a relationship, operation, or function that is analogous to such proportionality, satisfying additivity and homogeneity

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Local maximum

A maximum within a restricted domain, especially a point on a function whose value is greater than the values of all other points near it.

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Local minimum

A point on a graph (or its associated function) such that the points each side have a greater value even though another point exists with a smaller value.

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Logarithm

the exponent by which another fixed value, the base, must be raised to produce that number

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Manifold

a topological space that looks locally like the "ordinary" Euclidean space and is Hausdorff

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Map projection

any systematic method of transforming the spherical representation of parallels, meridians and geographic features of the Earth's surface to a nonspherical surface, usually a plane

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Chapter 1 - Functions and Models

Mathematical model

An abstract mathematical representation of a process, device or concept; it uses a number of variables to represent inputs, outputs and internal states, and sets of equations and inequalities to describe their interaction.

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Chapter 1 - Functions and Models

Mean

The average value.

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Meridian

an imaginary great circle on the Earth's surface, passing through the geographic poles

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Meromorphic

relating to or being a function of a complex variable that is analytic everywhere in a region except for singularities at each of which infinity is the limit and each of which is contained in a neighborhood where the function is analytic except for the singular point itself

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Momentum

(of a body in motion) the product of its mass and velocity

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Multivariable

concerning more than one variable

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Non-linear differential equation

nonlinear partial differential equation is partial differential equation with nonlinear terms

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Normal

a line or vector that is perpendicular to another line, surface, or plane

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Optimization

the design and operation of a system or process to make it as good as possible in some defined sense

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Origin

the point at which the axes of a coordinate system intersect

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Parallelogram

a convex quadrilateral in which each pair of opposite edges is parallel and of equal length

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Perpendicular

at or forming a right angle (to)

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Polar

of a coordinate system, specifying the location of a point in a plane by using a radius and an angle

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Polynomial

an expression consisting of a sum of a finite number of terms, each term being the product of a constant coefficient and one or more variables raised to a non-negative integer power

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Power law

any of many mathematical relationships in which something is related to something else by an equation of the form f(x) = a x^k

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Predator

any animal or other organism that hunts and kills other organisms (their prey), primarily for food

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Pressure

the amount of force that is applied over a given area divided by the size of this area

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Prey

a living thing that is eaten by another living thing

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Probability density function

any function whose integral over a set gives the probability that a random variable has a value in that set

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Programming languages

code of reserved words and symbols used in computer programs, which give instructions to the computer on how to accomplish certain computing tasks

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Proprietary

Manufactured exclusively by the owner of intellectual property rights (IPR), as with a patent or trade secret.

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Range

the set of values (points) which a function can obtain

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Real numbers

a value that represents a quantity along a continuous line

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Recurrence relation

an equation that recursively defines a sequence; each term of the sequence is defined as a function of the preceding terms

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Revolution

rotation: the turning of an object around an axis

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Rigid body

an idealized solid whose size and shape are fixed and remain unaltered when forces are applied; used in Newtonian mechanics to model real objects

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Roots

A zero (of a function).

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Scalar

a quantity that has magnitude but not direction; compare vector

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Scalar field

a function that assigns a scalar value to every point in space

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Scalar functions

any function whose domain is a vector space and whose value is its scalar field

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Scientific calculators

An electronic calculator that can handle trigonometric, exponential and often other advanced functions, and can show its output in scientific notation and sometimes in hexadecimal, octal or binary

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Secant

a straight line that intersects a curve at two or more points

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Secant lines

A line that (locally) intersects two points on the curve

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Sequence

an ordered list of objects

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Series

the sum of the terms of a sequence

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Set

a collection of distinct objects, considered as an object in its own right

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Sharpness

the fineness of the point a pointed object

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Slope

also called gradient; slope or gradient of a line describes its steepness

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Spherical coordinates

a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith

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Stochastic

Random, randomly determined

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Stokes' theorem

a theorem stating that says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative dω over the whole of Ω

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Summand

something which is added or summed

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Supercomputers

any computer that has a far greater processing power than others of its day; typically they use more than one core and are housed in large clean rooms with high air flow to permit cooling. Typical uses are weather forecasting, nuclear simulations and animations

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Surface area

the total area on the surface of a three-dimensional figure

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Surplus

specifically, an amount in the public treasury at any time greater than is required for the ordinary purposes of the government

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Symmetry

Exact correspondence on either side of a dividing line, plane, center or axis.

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Tangent

a straight line touching a curve at a single point without crossing it there

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Topology

a branch of mathematics studying those properties of a geometric figure or solid that are not changed by stretching, bending and similar homeomorphisms

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Torus

the standard representation of such a space in three-dimensional Euclidean space; a shape consisting of a ring with a circular cross-section; the shape of an inner tube or hollow doughnut

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Trajectory

the path of a body as it travels through space

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Transcendental

of or relating to a number that is not the root of any polynomial that has positive degree and rational coefficients

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Trapezoidal

in the shape of a trapezoid, or having some faces which have one pair of parallel sides

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Trigonometric

relating to the functions used in trigonometry: sin, cos, tan, csc, cot, sec

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Trigonometric functions

any function of an angle expressed as the ratio of two of the sides of a right triangle that has that angle, or various other functions that subtract 1 from this value or subtract this value from 1 (such as the versed sine)

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Variables

a quantity that may assume any one of a set of values

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Vector

a directed quantity, one with both magnitude and direction; the signed difference between two points

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Vector field

a construction in which each point in a Euclidean space is associated with a vector; a function whose range is a vector space

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Chapter 5 - Vector Calculus

Velocity

A vector quantity that denotes the rate of change of position with respect to time, or a speed with the directional component

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Volume

a unit of three-dimensional measure of space that comprises a length, a width and a height; measured in units of cubic centimeters in metric, cubic inches, or cubic feet in English measurement

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Chapter 2 - Applications of Integration

Zeno's dichotomy

That which is in locomotion must arrive at the half-way stage before it arrives at the goal.

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E, δ)-definition

a formalization of the notion of the limit of functions

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